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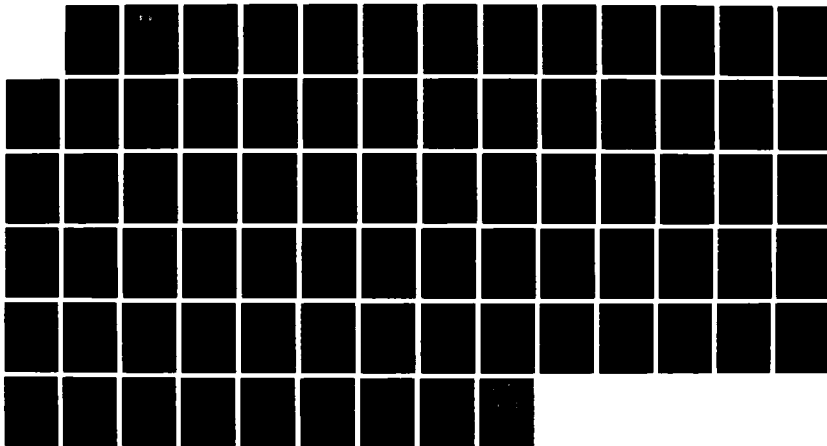
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PROMETHEUS INC.

FINAL REPORT

NULL STEERING APPLICATIONS OF POLYNOMIALS
WITH UNIMODULAR COEFFICIENTS

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PHASE I FINAL REPORT

ABSTRACT

Concerning adaptive array and null steering applications of polynomials with restricted coefficients, the basic mathematical question to consider in electronic beam steering, with a discrete array consisting of omni-directional elements spaced at equal increments along a straight line, is how coefficients of a polynomial may be chosen in a robust yet computationally efficient manner so as to arrive at a desired beam pattern. In numerous applications, these coefficients are required to satisfy certain restrictions, such as a bound on their dynamic range. Thus, particularly in null steering, it is often advantageous, or even necessary, for the shading coefficients to all have the same magnitude. Basic properties of such polynomials and their applications to beamforming are described.

A. STATEMENT OF WORK

Determine the robustness of the polynomials with unimodular coefficients introduced by the principal investigator in 1977.

Determine how to best introduce multiple nulls by employing the theory of polynomials with unimodular coefficients.

Develop adaptive array processing algorithms employing the concept of unimodular shading coefficients.

Determine to what extent coherent variation of shading coefficients can be controlled by restricting them to be unimodular.

Determine methods of choosing the best averaging process, as described in the Phase I Work Plan, for constructing polynomials with the required properties.

Solve the Littlewood and Erdős conjectures for F_n .

Construct actual polynomials which satisfy the Littlewood conjecture for G_n .

Develop a theory of interpolation by polynomials with unimodular coefficients and determine applications of these ideas to null steering and adaptive array processing.

Study the zeroes of these polynomials for both theoretical and applied purposes.

B. STATUS OF RESEARCH EFFORT

The following narrative describes the significant accomplishments, progress towards research objectives, new discoveries, and specific applications of the Phase I research carried out by Prometheus Inc.

I. INTRODUCTION

The fundamental purpose of our current research is to contribute to a deeper understanding of the properties and applications of polynomials with restricted coefficients, and in particular with coefficients of magnitude 1. Such understanding is crucial to the area of array design and to the construction of robust, computationally efficient adaptive array algorithms. This report, with its appendices, summarizes our progress to date.

The basic mathematical question to consider in electronic beam steering, with a discrete array consisting of omnidirectional elements spaced at equal increments along a line, is how coefficients of a polynomial may be chosen so as to arrive at a desired beam pattern. In numerous applications these coefficients are required to satisfy certain restrictions, such as a bound on their dynamic range. Here,

dynamic range refers to the ratio of the largest to the smallest magnitude. Thus, particularly in null steering, it is often advantageous, or even necessary, for the shading coefficients to all have the same magnitude.

Although the mathematical, statistical, and physical problems that arise in the consideration of array shading have been studied for roughly half a century, many interesting questions remain. This is true even for the "simple" case of the discrete line array referred to above. In this case, of course, letting n denote the number of elements, the pattern function $G(z)$ is a polynomial of degree $n-1$, z is a point on the unit circle, and the shading coefficients are just the n coefficients of this polynomial. An important reason for performing array shading is to shape the pattern function G so that it has low sidelobes and small beamwidth. As is well known, both of these quantities cannot be minimized simultaneously, and the choice of shading coefficients results in a tradeoff between these two desirable ends.

Electronic beam steering is another fundamental purpose of array shading, and it is this application that we address in this report. In addition to permitting the rotation of the main response axis of the pattern function, beam steering allows the simultaneous formation of a number of beams in different directions. In particular, if sources of interference lie at bearings different from that of the desired signal, then the signal-to-noise plus interference

ratio (SNIR) may be increased dramatically by directing nulls of the pattern function toward these interfering sources, in spite of the fact that the absolute power of the desired signal is thereby reduced. Adaptive techniques have been developed, by which array processing systems can electronically respond to an unknown interference environment. However, although the basic adaptive array principles have been known for some time, their application has been limited by hardware constraints and by the lack of sufficiently robust, real-time algorithms. New approaches to this latter consideration are described herein.

There are many cases when constraints must be placed upon the magnitudes of the coefficients of the pattern function. Thus, as explained by Hudson [13], when coefficients are implemented by attenuation they must be scaled so that the largest modulus is unity, since the amplitude gain for the desired transmission, and even the overall output signal-to-noise ratio (SNR), can be reduced by large coefficients. In discussing main-lobe constraints on optimal arrays, Hudson observes that when a main-lobe null is created very large shading coefficients are formed, resulting in enhanced output of uncorrelated noise. Hence, size restrictions on the coefficients are again required.

On the other hand, in a situation such as occurs in an adaptive radar receiver after clutter has decayed due to increasing range, so that there will be few and widely spaced target echoes of

minimal power compared to a steady jamming source, it is necessary to constrain the adaptive array so that the shading coefficients are prevented from falling to zero. A similar situation occurs in an adaptive antenna using the least mean square (LMS) algorithm, where the shading coefficients will decay to zero if either the signal level falls to 0, or if the reference signal is absent for some reason. One method of controlling this is to substitute the steered gradient system described by Griffiths for the reference signal LMS antenna, but this has the disadvantage of being very sensitive to errors in the assumed direction of the desired signal.

As mentioned earlier, another approach to these questions is to restrict the dynamic range of the shading coefficients. Although an informal rule of thumb for this range appears to be "2 and everyone is happy, 10 and some are happy, 100 and nobody is happy," a formal mathematical study of the relevant properties of polynomials, whose coefficients are thereby restricted, does not seem to have been previously undertaken. An important thrust of the research effort reported herein has been to initiate such a study, and to relate the large amount of work that has been accomplished by mathematicians, on polynomials with restricted coefficients, to the above applications. These efforts will continue in Phase II.

Furthermore, there is an intimate relationship between the engineering questions described above and several areas of classical mathematical analysis. Foremost among the problems of mutual interest

is the question of how close to constant the modulus of a polynomial can be along some curve, typically the unit circle. This is of deep concern to theoretical mathematical analysts because of the fundamental nature of polynomials and the simplicity and intrinsic beauty of the question, and it is equally important to engineers working in such fields as ϕ -design, adaptive beamforming and null steering, filter design, peak power limited transmitting, and the design of reflection phase gratings. This report describes our research into both aspects of this remarkable intertwining between the disciplines of pure mathematics and engineering.

II. MATHEMATICAL RESULTS

Concerning the purely mathematical aspects of our work, note that properties of polynomials with restricted coefficients have been the subject of much fruitful research in twentieth century mathematical analysis. Of particular interest have been polynomials with coefficients ± 1 or complex of modulus one. The study of such functions was apparently initiated by G.H. Hardy (see Zygmund [23, p.199]), and furthered by J.E. Littlewood, P. Erdős and others.

For the purposes of this discussion, it will be convenient to introduce the notation of Littlewood [17]. Thus, let F_n and G_n be, respectively, the class of all polynomials of the form

$$f(z) = \sum_{k=0}^n \pm 1 z^k, \quad g(z) = \sum_{k=0}^n \exp(a_k i) z^k,$$

where $|z|=1$ and the a_k are arbitrary real constants. Clearly the L^2 norm of g is $(n+1)^{1/2}$ for all $g \in G_n$ (and hence for all $f \in F_n \subset G_n$), and the question "how close can such a g come to satisfying

$$|g| = (n+1)^{1/2}.$$

has long been the object of intense study.

The first qualitative result concerning the above question for G_n was obtained by G.H. Hardy [23, p.199], who demonstrated the existence of a positive constant C and a sequence $\{g_n\}$, $g_n \in G_n$, satisfying $|g_n(z)| \leq C n^{1/2}$ for all n and z . The identical result for F_n was obtained by Shapiro [22] and published by Rudin [19]. Littlewood [16] conjectured that there exist positive constants A and B such that, for any n , there is an $f \in F_n$ ($g \in G_n$) satisfying

$$An^{1/2} \leq |f(z)| \leq Bn^{1/2} \quad (An^{1/2} \leq |g(z)| \leq Bn^{1/2})$$

for all z , while Erdős conjectured [11] that there is a positive constant C such that for $n \geq 2$, $\|g\|_\infty \geq (1+C)n^{1/2}$ for all $g \in G_n$ (and hence for all $f \in F_n$). Analogous conjectures for the L^p norms of $g \in G_n$ were settled in a series of papers by Beller and Newman ([1], [2], and [3]). Beller and Newman [4] also proved the Littlewood conjecture for polynomials whose coefficients have moduli bounded by 1, after observing that the proof of this result given by Clunie [10] depended upon an erroneous result of Littlewood. In [15] Körner was able to modify the results of Byrnes in [6] to prove the Littlewood conjecture for G_n , and then Kahane [14] showed that the Erdős conjecture is false for G_n . These conjectures for F_n remain unresolved.

One approach to the Erdős conjecture for polynomials in F_n is to consider their L^∞ norm. Our principal results in this regard are described in Appendix B. For polynomials in G_n , we have the following result:

Theorem 1. For each positive integer n , there is a sequence of coefficients $\{c_k\}_{k=0}^n$ such that all $|c_k| = 1$ and

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^n c_k e^{ik\theta} \right|^4 d\theta \leq (n+1)^2 + 4(n+1)^{3/2}.$$

Proof We show that in fact the Gauss coefficients,

$$c_k = e^{\frac{\pi i k^2}{n+1}},$$

satisfy the required property. Toward that end, note that

$$\sum_{k=0}^n c_k e^{ik\theta} \sum_{m=0}^n \bar{c}_m e^{-im\theta} = n+1 + \sum_{j \neq 0} \left(\sum_{m=0}^n c_{m+j} \bar{c}_m \right) e^{ij\theta}.$$

Therefore, by Parseval's Theorem, assuming for convenience that n is even,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^n c_k e^{ik\theta} \right|^4 d\theta &= (n+1)^2 + \sum_{j \neq 0} \left| \sum_{m=0}^n c_{m+j} \bar{c}_m \right|^2 \\ &= (n+1)^2 + 2 \sum_{j=1}^n \frac{\sin^2 \frac{j^2 \pi}{n+1}}{\sin^2 \frac{j \pi}{n+1}} \\ &= (n+1)^2 + 4 \sum_{j=1}^{n/2} \left(\frac{\sin \frac{j^2 \pi}{n+1}}{\sin \frac{j \pi}{n+1}} \right)^2 \\ &\leq (n+1)^2 + 4 \sum_{j=1}^{n/2} \min(j^2, (n+1)^2 / (2j)^2) \\ &\leq (n+1)^2 + 4(n+1)^{3/2}. \end{aligned}$$

where we have used the facts that

$$|\sin jx/\sin x| \leq j, \text{ and } |1/\sin x| \leq \pi/2x \text{ for } 0 < x < \pi/2.$$

This completes the proof of Theorem 1.

Another method of constructing polynomials with unimodular coefficients is to form a suitable weighted average of existing ones. For example, we may employ a slight variation of the basic construction in [6] as follows:

For each m , $0 \leq m \leq N^2-1$, and for $z = e^{2\pi i \theta}$, let

$$P_m(z) = P_m(\theta) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{2\pi i \left(\frac{jk}{N} + (j+kN) \frac{m}{N^2} \right)} z^{j+kN}.$$

Clearly, each P_m is a polynomial of degree N^2-1 with coefficients of modulus 1. Furthermore, it follows from [6] that, for a suitable small positive ϵ (i.e., of order N^{-2}), $|P_m(\theta)|$ is essentially flat for

$$\epsilon - m/N^2 \leq \theta \leq 1 - \epsilon - m/N^2.$$

Now define $P^*(\theta)$ by

$$P^*(\theta) = \sum_{m=0}^{N^2-1} z^{-mN^2} P_m(\theta).$$

P^* is a polynomial of degree N^2-1 with coefficients of modulus 1.

Also, by writing

$$P^*(\theta) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{2\pi i \frac{jk}{N}} z^{j+kN} \frac{1 - z^{N^2}}{1 - e^{2\pi i \frac{j+kN}{N}} e^{2\pi i N^2 \theta}}$$

and letting

$$\tilde{\theta} = -N^{-2}(A+BN+CN^2) \text{ for } 0 \leq A \leq N-1, 0 \leq B \leq N-1, 0 \leq C \leq N^2-1,$$

it is seen that

$$P^*(\tilde{\theta}) = N^2 e^{2\pi i \left(\frac{A^2}{N} - (A+BN) \frac{A+BN+CN^2}{N^2} \right)}$$

so that $|P^*(\tilde{\theta})| = N^2$.

In addition, the essential flatness of $|P^*(\tilde{\theta})|$ in the interval $E \leq \theta \leq 1-E$, where now E is of order N^{-2} , follows as before. However, numerical evidence suggests that $P^*(2^{-1}N^{-2}) = O(1)$, a similar situation to that which occurred with the original polynomials [6]. This being the case, P^* is not quite a Kahane-type polynomial, as we had originally hoped.

Note, however, that the above method of constructing $P^*(\theta)$ can also be employed to create new flat spectrum sequences, which are periodic sequences $\{a_n\}_{n=0}^{\infty}$ with the property that their discrete Fourier transform (DFT) has a power spectrum consisting of a very small number (usually 1 or 2) of distinct values. This is because the DFT can be thought of as the values of the polynomial

$$P(z) = \sum_{k=0}^{n-1} a_k z^k,$$

where n is the period, at the n -th roots of unity. Our construction yields polynomials whose spectra are essentially flat at almost all points of the unit circle, not just at the roots of unity. Observe that flat spectrum sequences constructed in this manner satisfy the additional property that all of the terms of the original sequence have the same magnitude. Applications of these concepts to notch filtering and communications are discussed elsewhere in this report.

Another method of viewing these questions is in the context of interpolation problems. As noted earlier, for any $P \in G_n$, the Parseval Theorem implies that the L^2 norm of P on the unit circle C is $(n+1)^{1/2}$. Furthermore, since $|P(z)|^2$ is expressible as $z^{-n}Q(z)$, where Q is of degree $2n$, there can be, at most, $2n$ distinct points z_k where

$$|P(z_k)| = (n+1)^{1/2}.$$

Let us call such a set of points an L^2 Interpolating Set for P . A natural question is which, if any, subsets of C consisting of $2n$ points can be an L^2 interpolating set for some P of the required form.

In its full generality, this question appears to be quite difficult. For $n=1$, it is trivial to show that $S=\{a,b\}$ is an L^2 Interpolating Set if and only if $b=-a$. For arbitrary n , observe that

for $S = \{z_k\}_{k=0}^{2n}$ to be an L^2 Interpolating Set, the coefficients of P must be chosen so that

$$Q(z) - (n+1)z^n = \alpha \prod_{k=1}^{2n} (z - z_k),$$

where α is a constant of modulus one. Furthermore, the coefficient of z^n on the left side of this equation vanishes, so the same must be true on the right side. Clearly, this will be a very rare occurrence, so that most sets will not be L^2 Interpolating Sets. In fact, it is not at all obvious that for $n > 1$, there exist any L^2 Interpolating Sets. Thus far, we are only able to show that if S is to be such a set, its elements cannot be too close to each other. More precisely,

Theorem 2 For any n , there is an $\epsilon > 0$ such that no S of the form

$$S = \{e^{i\theta_k}\}_{k=1}^{2n}, \text{ with } |\theta_k| \leq \epsilon \text{ for } 1 \leq k \leq 2n,$$

is an L^2 Interpolating Set for any $P \in G_n$.

Proof of Theorem 2. Assume the contrary. Fix n . Then, for any $\epsilon > 0$, there is a set

$$S = S(\epsilon, n) = \{e^{i\theta_k}\}_{k=1}^{2n}, \text{ with } |\theta_k| \leq \epsilon \text{ for } 1 \leq k \leq 2n,$$

such that S is an L^2 Interpolating Set for some P , say

$$P_{\epsilon, n}(z) = \sum_{k=0}^n a_{\epsilon, k} z^k, \text{ with all } |a_{\epsilon, k}| = 1.$$

Choose a sequence of positive ϵ 's, say $\{\epsilon_j\}_{j=1}^{\infty}$, approaching 0.

For each k , $0 \leq k \leq n$, the sequence $\{a_{\epsilon_j, k}\}_{j=1}^{\infty}$

is bounded, and all terms of each of these $n+1$ sequences have modulus 1. By the standard method of choosing a convergent subsequence for one k at a time, we can find a strictly increasing sequence of positive integers

$$\{m_j\}_{j=1}^{\infty}, \text{ and a set } \{a_k\}_{k=0}^n$$

of complex numbers all of modulus 1, such that

$$\{a_{\epsilon_{m_j}, k}\}_{j=1}^{\infty} \text{ converges to } a_k \text{ for every } k, 0 \leq k \leq n.$$

Since $|P_{\epsilon_{m_j}}(e^{i\theta})| - (n+1)^{1/2}$ can't change sign for

$$|\epsilon_{m_j}| < \theta \leq \pi,$$

we can assume, by taking another subsequence if necessary, that either

$$|P_{\epsilon_{m_j}}(e^{i\theta})| - (n+1)^{1/2}$$

is always positive or always negative for

$$|\epsilon_{m_j}| < \pi.$$

Suppose the former (the argument being the same in the latter case), and define

$$P_0(z) = \sum_{k=0}^n a_k z^k.$$

Clearly $\{P_{\epsilon_{m_j}}(z)\}_{j=1}^{\infty}$ converges uniformly to $P_0(z)$ on $|z| = 1$,

so that $|P_0(e^{i\theta})| \geq (n+1)^{1/2}$ for $0 < \theta \leq 2\pi$.

Since the L^2 norm of P_0 is $(n+1)^{1/2}$, this is impossible, and the proof of Theorem 2 is complete.

Also of interest is the locations of the zeroes of polynomials with unimodular coefficients. This is directly related to many other problems discussed herein and has obvious importance in the choice of pattern functions for null steering. To quantify this question, let $r_j \cdot e^{i\alpha_j}$, $1 \leq j \leq n$, be the zeroes of $P_n \in G_n$, normalize P_n so that the coefficient of z^n is 1, and define

$$\mathcal{J}_n = \max_j \min |1 - r_j| \text{ and } \mathcal{J}_{n,a} = \max \left(\sum_{j=1}^n |1 - r_j|^2 \right)^{1/2},$$

where the maximum is taken over all such $P_n(z)$.

Since any $P_1(z) = z - e^{i\alpha_1}$ for some real α_1 , it is obvious that

$$\lambda_1 = \lambda_{1,a} = 0.$$

Considering the case $n=2$,

$$P_2(z) = z^2 - (r_1 e^{i\alpha_1} + r_2 e^{i\alpha_2})z + r_1 r_2 e^{i(\alpha_1 + \alpha_2)},$$

so that

$$r_1 r_2 = |r_1 e^{i\alpha_1} + r_2 e^{i\alpha_2}| = 1.$$

Assume that $r_1 \geq 1$. Since

$$1 = |r_1 e^{i\alpha_1} + r_2 e^{i\alpha_2}| \geq r_1 - r_2 = r_1 - 1/r_1,$$

the maximum value for $r_1 - 1/r_1$ (hence the maximum value for $r_1 - 1$) is achieved when $r_1 - 1/r_1 = 1$, or

$$r_1 = \frac{1+\sqrt{5}}{2}, \text{ and } r_2 = \frac{2}{1+\sqrt{5}}.$$

In this case,

$$r_1 - 1 = \frac{\sqrt{5}-1}{2} \text{ and } 1 - r_2 = \frac{\sqrt{5}-1}{1+\sqrt{5}} < \frac{\sqrt{5}-1}{2},$$

so that

$$\lambda_2 = \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{3-\sqrt{5}}{2}.$$

Also,

$$\lambda_{2,2}^2 = \max_r ((r-1)^2 + (1-1/r)^2).$$

By an elementary calculus argument, it is seen that this maximum occurs for $r=(\sqrt{5}+1)/2$. Thus,

$$\lambda_{2,2} = \sqrt{5-2\sqrt{5}}.$$

We leave as an open question the behavior of other values of λ_n and $\lambda_{n,q}$.

The final mathematical question which we examine here is one which is particularly important for applications. Namely, how robust are polynomials with unimodular coefficients? Thus, expressing $P \in G_n$ as

$$P(z) = \sum_{k=0}^n A_k z^k, \quad |A_k| = 1,$$

consider that the coefficients A_k are not fixed quantities but may be written as $A_k + a_k$, where $a_k = a_k(\omega)$ is a random variable representing the coefficient error. Hence, the polynomial becomes

$$Q(z) = \sum_{k=0}^n A_k z^k + \sum_{k=0}^n a_k z^k,$$

where the final sum represents an error polynomial whose behavior is to be analyzed to determine the robustness of a system employing these functions.

Since the class F_n of polynomials of the form

$$P(z) = \sum_{k=0}^n \pm 1 z^k$$

is of particular interest, let us begin by examining error polynomials of the form

$$f(z) = \sum_{k=0}^n a_k z^k,$$

where a_k are real random variables which we assume to be independent and normally distributed with mean 0 and variance 1, to be written $N(0,1)$. Later, we shall indicate how to generalize our work to the case of other random coefficients and, finally, to complex random coefficients.

Let us rewrite the error polynomial as

$$f(t) = \sum_{k=0}^n a_k(\omega) e^{i k t},$$

where the a_k are independent real random variables which are $N(0,1)$ and $t \in [0, 2\pi]$. We begin by considering the real part of $f(t)$, which is

$$\operatorname{Re} f(t) = P(t, \omega) = \sum_{k=0}^n a_k(\omega) \cos kt.$$

In estimating the effects of coefficient inaccuracy in transversal filtering, Gersho, Gopenath and Odlyzko [12] analyze similar error polynomials. In comparing various performance measures, they conclude that the most appropriate measure is the maximum deviation in frequency response magnitude from the desired values over a particular frequency band. Their analyses and conclusions apply in our case and lead us to estimate the random variable

$$M^*(\omega) = \sup_{t \in [0, 2\pi)} P(t, \omega).$$

Our methods, which differ from those of Gersho et al, are based on pioneering work in random polynomials done by Salem and Zygmund [20]. They yield good estimates without unknown constants and are computationally simple.

Let us now outline the procedure, beginning with two lemmas which give estimates of the size of $P(t, \omega)$.

Lemma 1. $E(e^{\lambda P(t, \omega)}) \leq e^{\frac{\lambda^2}{2}(n+1)}$, where E represents expectation.

Proof. It is well known that for $a_k \sim N(0, 1)$, the moment generating function of a_k is given by $E(e^{\gamma a_k}) = e^{\gamma^2/2}$.

Then

$$\begin{aligned}
E(e^{\lambda P(t, \omega)}) &= E(e^{\sum_{k=0}^n \lambda a_k(\omega) \cos kt}) = E(\prod_{k=0}^n e^{\lambda a_k(\omega) \cos kt}) \\
&= \prod_{k=0}^n E(e^{\lambda \cos kt a_k(\omega)}) \\
&= \prod_{k=0}^n e^{\frac{\lambda^2 \cos^2 kt}{2}} \leq \prod_{k=0}^n e^{\frac{\lambda^2}{2}} = e^{\frac{\lambda^2}{2}(n+1)},
\end{aligned}$$

where the second line follows by the independence of the a_k 's. This completes the proof of Lemma 1.

Lemma 2. Define $M(\omega) = \sup_{t \in [0, 2\pi)} |P(t, \omega)|$.

Then there exists, for each ω , a random set $I(\omega) \subseteq [0, 2\pi)$ of Lebesgue measure at least $1/n$ such that

$$|P(t, \omega)| \geq M(\omega)/2 \text{ for each } t \in I(\omega).$$

Proof. By Bernstein's theorem,

$$\|P'\|_{\infty} = \sup_{t \in [0, 2\pi)} \left| \frac{\partial P}{\partial t}(t, \omega) \right| \leq nM(\omega).$$

Since $P(t, \omega)$ is continuous for each ω , there exists $t_0 \in [0, 2\pi)$ such that $M(\omega) = \pm P(t_0, \omega)$. Thus, for fixed ω ,

$$|P(t, \omega) - P(t_0, \omega)| \leq |t - t_0| \|P'\|_{\infty} \leq |t - t_0| n M(\omega).$$

If $M(\omega) = +P(t_0, \omega)$, $|P(t, \omega) - M(\omega)| \leq M(\omega)/2$ for $|t - t_0| n \leq 1/2 \pmod{2\pi}$.

If $M(\omega) = -P(t_0, \omega)$, $|P(t, \omega) + M(\omega)| \leq M(\omega)/2$ for $|t - t_0|n \leq 1/2 \pmod{2\pi}$.

Thus, $|P(t, \omega)| \geq M(\omega)/2$ for $t \in [t_0 - 1/2n, t_0 + 1/2n] \pmod{2\pi}$.

and Lemma 2 is proven.

We combine the above lemmas to obtain the principal estimation.

Theorem 3. $E(e^{M\lambda/2}) \leq 4\pi n e^{\frac{\lambda^2}{2}(n+1)}$

Proof.
$$E(e^{M\lambda/2}) = \int_{\Omega} e^{M(\omega)\lambda/2} dP(\omega)$$
$$\leq \int_{\Omega} e^{M(\omega)\lambda/2} \left(\int_0^{2\pi} n 1_{I(\omega)}(t) dt \right) dP(\omega),$$

where we have used Lemma 2 with

$$1_{I(\omega)}(t) = \begin{cases} 1 & \text{if } t \in I(\omega) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore,

$$E(e^{M\lambda/2}) \leq n \int_{\Omega} \left(\int_0^{2\pi} e^{M(\omega)\lambda/2} 1_{I(\omega)}(t) dt \right) dP(\omega)$$
$$\leq n \int_{\Omega} \left(\int_0^{2\pi} [e^{\lambda P(t, \omega)} + e^{-\lambda P(t, \omega)}] dt \right) dP(\omega),$$

because

$$e^{M(\omega)\lambda/2} 1_{\{M(\omega) \geq t\}} \leq e^{\lambda^2 t/2} + e^{-\lambda^2 t/2}.$$

Now use Fubini's Theorem to interchange the order of integration, and apply Lemma 1 to obtain

$$E(e^{M\lambda/2}) \leq \int_0^{2\pi} 2e^{\frac{\lambda^2}{2}(n+1)t} dt = 4\pi n e^{\frac{\lambda^2}{2}(n+1)},$$

and we have Theorem 3.

To obtain our estimate of the size of M , we prove the following:

Theorem 4. $P\{M \geq \alpha\} \leq 4\pi n e^{-\alpha^2/8(n+1)}$ for $\alpha \geq 0$.

Proof. Let $\lambda > 0$ and use the Markov Inequality and Theorem 3 to obtain

$$P\{M \geq \alpha\} = P(e^{M\lambda/2} \geq e^{\alpha\lambda/2}) \leq E(e^{M\lambda/2})/e^{\alpha\lambda/2} \leq 4\pi n e^{\frac{\lambda^2}{2}(n+1) - \frac{\alpha\lambda}{2}}.$$

To get the best bound for a given α , we employ elementary calculus to minimize

$$h(\lambda) = \lambda^2(n+1)/2 - \alpha\lambda/2,$$

yielding $\lambda_c = \alpha/2(n+1)$. Substituting λ_c into the right side above yields Theorem 4.

We can now obtain similar estimates for

$$N(w) = \sup_{t \in [0, 2\pi)} |f(t, w)|, \text{ where}$$

$$f(t, w) = \sum_{k=0}^n a_k(w) e^{ik t}$$

and the a_k are independent $N(0, 1)$.

$$\text{Theorem 5. } P\{N(w) \geq \alpha\} \leq 8T \text{Ine}^{-\alpha^2/16(n+1)}.$$

Proof. Let $\text{Re } f = \sum_{k=0}^n a_k \cos kt$ and $\text{Im } f = \sum_{k=0}^n a_k \sin kt$. We have shown in Theorem 4 that

$$P\left\{\sup_{t \in [0, 2\pi)} |\text{Re } f(t, w)| \geq \alpha / \sqrt{2}\right\} \leq 4T \text{Ine}^{-\alpha^2/16(n+1)}.$$

By applying the analysis in Lemmas 1 and 2 and Theorems 3 and 4 to $\text{Im } f$, we obtain

$$P\left\{\sup_{t \in [0, 2\pi)} |\text{Im } f(t, w)| \geq \alpha / \sqrt{2}\right\} \leq 4T \text{Ine}^{-\alpha^2/16(n+1)}.$$

Since $|f(t, w)|^2 = |\text{Re } f(t, w)|^2 + |\text{Im } f(t, w)|^2$, we have

$$\left\{\sup_{t \in [0, 2\pi)} |f(t, w)|^2 \geq \alpha^2\right\} \subseteq \left\{\sup_{t \in [0, 2\pi)} |\text{Re } f(t, w)|^2 \geq \alpha^2/2\right\} \cup \left\{\sup_{t \in [0, 2\pi)} |\text{Im } f(t, w)|^2 \geq \alpha^2/2\right\}.$$

Therefore,

$$P\{N \geq \alpha\} \leq P\left\{\sup_{t \in [0, 2\pi)} |\text{Re } f(t, w)| \geq \alpha / \sqrt{2}\right\} + P\left\{\sup_{t \in [0, 2\pi)} |\text{Im } f(t, w)| \geq \alpha / \sqrt{2}\right\} \leq 8T \text{Ine}^{-\alpha^2/16(n+1)},$$

completing the proof of Theorem 5.

We now choose α to obtain suitable explicit bounds. If we put

$$\alpha = 4\sqrt{(n+1)(\log 8\pi n + 4 \log \log n)}, \text{ then}$$

$$P\{N \geq \alpha\} \leq 1/(\log n)^4,$$

which improves an estimate in Theorem 1 of [12].

If $\alpha = 4\sqrt{(n+1) \log (8\pi n^2)}$, then

$$P\{N \geq \alpha\} \leq 1/n.$$

In this way, estimates of any desired precision may be obtained.

When coefficient error is due to digital round-off, it is reasonable to assume that the coefficients a_k are uniformly distributed on the interval $[-1/2, 1/2]$, denoted $a_k \sim U[-1/2, 1/2]$. The following result gives the analogue of Theorem 5 in this case.

Theorem 6. Let

$$f(t, \omega) = \sum_{k=0}^n a_k(\omega) e^{ikt}, \quad t \in [0, 2\pi),$$

where the a_k are independent random variables which are $U[-1/2, 1/2]$.

Then

$$P\{N \geq \alpha\} \leq 8\pi n e^{-\alpha^2/4(n+1)}.$$

Proof. A direct calculation shows that

$$E(e^{\lambda a_k}) = \begin{cases} (e^{\frac{\lambda}{2}} - e^{-\frac{\lambda}{2}})/\lambda & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0 \end{cases}.$$

By simple use of the exponential series, we have

$$E(e^{\lambda a_k}) = 1 + \frac{(\lambda/2)^2}{3!} + \frac{(\lambda/2)^4}{5!} + \dots = \sum_{k=0}^{\infty} \frac{(\lambda/2)^{2k}}{(2k+1)!} \leq \sum_{k=0}^{\infty} \frac{[(\lambda/2)^2]^k}{2^k k!},$$

using $2^k k! \leq (2k)! \leq (2k+1)!$

Finally,

$$E(e^{\lambda a_k}) \leq \sum_{k=0}^{\infty} \frac{(\lambda^2/8)^k}{k!} = e^{\lambda^2/8}.$$

We now follow the steps in the procedure for normal coefficients to obtain similar results to Theorem 5. First consider

$$P(t, w) = \sum_{k=0}^n a_k \cos kt \text{ and } M(w) = \sup_{t \in [0, 2\pi]} |P(t, w)|.$$

Following Lemma 1, we obtain

$$E(e^{\lambda P(t, w)}) \leq e^{\frac{\lambda^2}{8}(n+1)}$$

and from Lemma 2,

$$E(e^{\frac{\lambda}{T} M}) \leq 4T \text{In} e^{\frac{\lambda^2}{8}(n+1)}.$$

As in Theorem 4, we have

$$P\{M \geq \alpha\} \leq 4T \text{In} e^{-\alpha^2/2(n+1)}.$$

We obtain a similar bound for the maximum of the imaginary part of $f(t, w)$, and we combine these as in Theorem 5 to obtain

$$P\{N \geq \alpha\} \leq P\left\{\sup_{t \in [0, 2\pi]} |\text{Re } f| \geq \alpha / \sqrt{2}\right\} + P\left\{\sup_{t \in [0, 2\pi]} |\text{Im } f| \geq \alpha / \sqrt{2}\right\} \leq 8T \text{In} e^{-\alpha^2/4(n+1)},$$

which completes the proof of Theorem 6.

Remark. We may obtain specific estimates for $P\{N \geq \alpha\}$ in the uniform case just as we did for the normal case after Theorem 5. For example, setting

$$\alpha = 2\sqrt{(n+1) \log(8T \ln^2)}, \text{ we obtain } P(N \geq \alpha) \leq 1/n.$$

We are now prepared to give estimates on the error polynomials obtained from polynomials with unimodular coefficients. These error polynomials are of the form

$$f(t, w) = \sum_{k=0}^n a_k e^{i k t},$$

where the a_k are independent complex random variables. In other words, $a_k = \alpha_k + i\beta_k$ where α_k and β_k are independent real random variables. We state the main Theorem.

Theorem 7. Let $N(w) = \sup_{t \in [0, 2\pi]} |f(t, w)|$, where

$$f(t, w) = \sum_{k=0}^n a_k(w) e^{i k t}$$

and the $a_k = \alpha_k + i\beta_k$ are independent random variables. We suppose that the real and imaginary components of each a_k , namely α_k and β_k , are independent. Then

- (a) If α_k and β_k are $N(0, 1)$, $P\{N \geq \alpha\} \leq 16T \ln e^{-\alpha^2/64(n+1)}$
 (b) If α_k and β_k are $U[-1/2, 1/2]$, $P\{N \geq \alpha\} < 16T \ln e^{-\alpha^2/16(n+1)}$.

Proof. Let us write

$$f(t, w) = f_1(t, w) + i f_2(t, w), \text{ where}$$

$$f_1(t, w) = \sum_{k=0}^n \alpha_k e^{i k t} \text{ and } f_2(t, w) = \sum_{k=0}^n \beta_k e^{i k t}.$$

Consider the case where α_k, β_k are independent $N(0,1)$. If

$$N_i(w) = \sup_{t \in [0, 2\pi]} |f_i(t, w)|, \quad i=1,2, \text{ then}$$

$$\{|f(t, w)| \geq \alpha\} \subseteq \{|f_1(t, w)| \geq \alpha / 2\} \cup \{|f_2(t, w)| \geq \alpha / 2\},$$

since $|f| \leq |f_1| + |f_2|$. Therefore,

$$P\{N(w) \geq \alpha\} \leq P\{N_1(w) \geq \alpha / 2\} + P\{N_2(w) \geq \alpha / 2\} = 2P\{N_1(w) \geq \alpha / 2\}.$$

Applying Theorem 6 yields

$$P\{N(w) \geq \alpha\} \leq 16T \exp \left(-\frac{\alpha^2}{64(n+1)} \right),$$

which is (a). A similar procedure applied to the uniform case gives (b), and we have Theorem 7.

In summary, we have obtained easily calculated and flexible upper bounds on the maximum deviation caused by random error polynomials. These bounds offer estimates of the robustness of a discrete array of omnidirectional elements.

III. APPLICATIONS

As mentioned in the introduction, applications of polynomials with restricted coefficients abound in the engineering world. Those which we focus on herein include null steering, adaptive beamforming, notch filtering, peak power limited transmitting, and the synthesis of low peak-factor signals and flat spectrum sequences.

In Appendix C, several new designs of analytic null steering algorithms for linear arrays are described. Two of them, the \mathcal{L} -Technique and the Positive Coefficient Model, allow for placing an arbitrary number of nulls in arbitrary directions, while maintaining main beam and sidelobe level control. A method of incorporating these deterministic null steering techniques into existing adaptive algorithms is proposed. The resulting Direct Adaptive Nulling System offers the possibility of significant increases in array performance at very little cost. This possibility will be investigated in full detail in Phase II.

A major reason for combining deterministic methods with existing techniques is that arrays must ordinarily deal with significant random noise. In these cases, one has no a priori information about the direction or nature of such unwanted signals. Thus, in such applications, as well as in cases where advance knowledge of jammer characteristics is lacking, indirect statistical methods are unavoidable, although their efficiency may be greatly increased by combining them with analytic approaches.

There exist applications, however, where much is known in advance about the characteristics of both the desired signals and the

undesired noise. This is especially true where one has control of the generation of these waveforms. Thus, in the case where one system is producing both offensive signals (i.e., searching for and homing in on targets) and defensive signals (i.e., identifying and tracking incoming weapons), so that mutual interference becomes a predominant concern, the problem is almost exclusively deterministic in nature. In such cases, robust and computationally efficient analytic algorithms controlling both the individual performance of the offensive and defensive signals and the interactive jamming between them are crucial to mission success. The application to such cases of the new deterministic null steering algorithms developed by Prometheus will be analyzed in detail during Phase II.

A related problem is the determination of optimal shading coefficients for a conformal array. As is well known, using various measures of optimality, this is a computational problem of order n^3 , where n is the number of array elements. Thus, the computational load will be reduced by a factor of 8 if the coefficients may be restricted to be real. Circumstances where this occurs are described in Appendix D. A different method of improving computational efficiency, namely a convex programming approach, will be an important focus of our Phase II research.

Another interesting application of our concepts is to notch filters. Appendix E describes a nearly ideal notch filter employing coefficients of equal magnitude. Applications to the design of transmitting antenna arrays are discussed briefly. The construction is based upon earlier work of the author involving polynomials with

restricted coefficients. The fundamental idea employed in Appendix E to construct a notch filter with a single notch may be combined with the concept of an n -nomial [5] to produce nearly ideal filters with multiple notches. Furthermore, as noted elsewhere, zero coefficients do not affect the dynamic range, so that these multi-notch filters maintain the property of having unit dynamic range. Details of this new construction and experimental results employing it will be developed during Phase II.

In addition to their use in the construction of notch filters, Byrnes Polynomials [6,14,15] have potential applications to the design of peak power limited transmitters and the synthesis of low peak-factor signals and flat spectrum sequences. In transmitter design, for example, one is often faced with a peak power constraint. Under various conditions, the transmitter output may be modeled as a polynomial. Here the maximum modulus of the polynomial on the unit circle represents the peak power, while the L^2 norm of the polynomial is the average power. Thus the classical engineering problem of minimizing the peak to average ratio becomes the mathematical question of minimizing the ratio of the sup norm to the L^2 norm of a polynomial on the unit circle.

In the trivial case where one frequency is to be transmitted (ie, the polynomial can be a monomial), clearly the ideal value 1 for the peak to average ratio is achieved, and the polynomial is indeed of constant modulus on the unit circle. For the more interesting and practical case of transmitting many linearly increasing frequencies, it is usually desired to transmit each

frequency at the same power, which should be as large as possible. As the power of each individual frequency is represented by the modulus of the corresponding coefficient, the mathematical question naturally arises of how close to constant the modulus of a polynomial with equimodular coefficients can be on the unit circle.

More precisely, if n pure tones are transmitted with frequencies of the form $f_0 + k\Delta$, where f_0 is the fundamental frequency and Δ is the increment, then the waveform is

$$x(t) = \sum_{k=0}^{n-1} A_k \cos [2\pi(f_0 + k\Delta)t + \theta_k] \\ = |S(t)| \cos[\arg S(t) + 2\pi f_0 t].$$

Here, $S(t) = \sum_{k=0}^{n-1} A_k e^{i\theta_k} e^{i2\pi k\Delta t}$, θ_k = phase and A_k = power in k^{th} tone.

As mentioned, almost always all frequencies are transmitted with equal power, so that $A_k = 1$. To minimize the peak power of $x(t)$, the maximum (over t) of $|x(t)|$ must be minimized (over θ_k). It is relatively straightforward to see that the exact problem is to obtain

$$\min_{\theta_k} \max_t \left| \sum_{k=0}^{n-1} e^{i\theta_k} e^{i2\pi k\Delta t} \right|,$$

a job which is performed by the Byrnes polynomials [5] in nearly ideal fashion.

The adaptation of such polynomials to these problems is important, since in applications like the Link 11 Communications System, the average power is usually maintained at one tenth or less of its theoretical ideal to prevent transmitter overload. Employing concepts such as those described above should yield a significant reduction in the peak-to-average ratio, thereby allowing a large

increase in average power, hence a more efficient communications system. These considerations also show that the Byrnes construction has direct application to the synthesis of low peak-factor signals.

Now consider the problem of designing a flat spectrum sequence $\{a_k\}_{k=0}^{\infty}$, as defined on page 10. These sequences have direct use in such diverse areas as concert hall acoustics, the quieting of an object's response to radar and active sonar, and speech synthesis. Schroeder [21] presents many of the fascinating details of these applications.

As we observed earlier, the DFT can be thought of as the values of the polynomial

$$P(z) = \sum_{k=0}^{n-1} a_k z^k,$$

where n is the period, at the n -th roots of unity. The Byrnes construction [6] yields polynomials whose spectra are essentially flat at almost all points of the unit circle, not just at the roots of unity. Furthermore, they have the additional property that all of the terms of the original sequence, $\{a_k\}$, have the same magnitude. Applications of these concepts to notch filtering and communications are discussed elsewhere in this report.

In our final application, we have begun to exploit the great success of J.P. Kahane [14] in solving the Littlewood conjecture. As we note in part II, Kahane showed that there indeed exist polynomials with unimodular coefficients whose modulus is essentially constant on the unit circle. It is our opinion that the breakthrough of Kahane was due to the ingenious use of randomness and probability in his

construction. Behind his and previous approaches was the idea of Gauss, viz. the "Gauss Sums." To put it quite simply, we feel that Littlewood's problem was vanquished by the "equation"

$$\text{Kahane} = \text{Gauss Sums} + \text{Probabilistic Choices}.$$

Our idea is to exploit the Kahane breakthrough by developing methods to judiciously make the "Probabilistic Choices" referred to above, and thereby convert Kahane's "randomized" proof into a constructive one. This would not only result in exciting new mathematics, but would also be directly applicable to several important engineering problems. In addition to the areas of peak power limited transmitting and flat spectrum sequences discussed earlier, such polynomials would find immediate use in the design of reflection phase gratings, and therefore be employable in solving concert hall acoustics problems and in quieting the response of an object to sonar or radar. Another potential application of this "educated randomness" construction is in the synthesis of multielement omnidirectional beam patterns.

In the concert hall acoustics application of reflection phase gratings, it is desired to design the ceiling so that sound is widely scattered except in the specular direction. As described earlier and in Appendix E, in the context of notch filter design, the Byrnes polynomials [6] place a null in any given direction while the coefficients maintain their other desirable properties of being both flat spectrum and low correlation sequences. Thus, they might even be preferable to the Kahane polynomials in this context. This also appears to apply to monostatic radar, where the null would be placed

in the direction of the radar. For bistatic radar, on the other hand, the receiver direction is often unknown. Thus, if a construction based upon the Kahane polynomials could be employed, radar energy would be reflected equally in all directions, thereby reducing the probability that there would be enough energy reflected in any particular direction to enable detection. A possible undersea application of these ideas occurs in the design of baffles used to quiet machinery noise from submarines, in an attempt to prevent the noise from escaping the hull. Note that our constructions would complement the coatings that are already in use, or being designed, to attack these problems, since these coatings provide uniform attenuation. Furthermore surface structures based upon the Byrnes polynomials would have the highly diffusing property over a large set of frequencies. It is not yet clear whether the Kahane polynomials also yield this important property. The design of two-dimensional arrays so that energy may be scattered with equal intensity over the solid angle is also of considerable interest. It appears that a straightforward product formulation gives the desired results for the Byrnes polynomials, but the situation is not so clear for Kahane polynomials. Our Phase II research will focus upon the many fascinating questions raised in this final paragraph.

BIBLIOGRAPHY

1. Beller, E., Polynomial Extremal Problems in L^p , Proc. Amer. Math. Soc. 30 (1971), 249-259.
2. Beller, E. and Newman, D.J., An L^1 Extremal Problem for Polynomials, Proc. Amer. Math. Soc. 29 (1971), 474-481.
3. Beller, E. and Newman, D.J., An Extremal Problem for the Geometric Mean of Polynomials, Proc. Amer. Math. Soc. 39 (1973), 313-317.
4. Beller, E. and Newman, D.J., The Minimum Modulus of Polynomials, Proc. Amer. Math. Soc. 45 (1974), 463-465.
5. Byrnes, J.S., L^2 Approximation with Trigonometric n-omials, J. of Approximation Theory 9 (1973), 373-379.
6. Byrnes, J.S., On Polynomials with Coefficients of Modulus One, Bull. London Math. Soc. 9 (1977), 171-176.
7. Byrnes, J.S., The Minimax Optimization of an Antenna Array Employing Restricted Coefficients, Scientia (to appear, 1987).
8. Byrnes, J.S. and Newman, D.J., Null Steering Employing Polynomials with Restricted Coefficients, (submitted).
9. Byrnes, J.S., A Notch Filter Employing Coefficients of Equal Magnitude, (submitted).
10. Clunie, J.C., The Minimum Modulus of a Polynomial on the Unit Circle, Quart. J. of Math. (Oxford), 10 (1959), 95-98.
11. Erdős, Paul, Some Unsolved Problems, Mich. Math. J. 4(1957), 291-300.
12. Gersho, Gopenath, and Odlyzko, Coefficient Inaccuracy in Transversal Filtering, The Bell System Technical Journal, Vol. 58, No. 10, 1979, 2301-2316.
13. Hudson, J.E., Adaptive Array Principles, Peter Peregrinus Ltd., New York, 1981.
14. Kahane, Jean-Pierre, Sur les Polynomes à Coefficients Unimodulaires, Bull. London Math. Soc. 12 (1980), 321-342.
15. Körner, T.W., On a Polynomial of Byrnes, Bull. London Math Soc. 12 (1980), 219-224.
16. Littlewood, J. E., On the Mean Values of Certain Trigonometric Polynomials, II, Illinois J. Math. 6 (1962), 1-39.

17. Littlewood, J. E., On Polynomials $\sum \pm 1 z^m$, $\sum \alpha_m i z^m$, $z=e^{i\theta}$, J. Lon. Math. Soc. 41 (1966), 367-376.
18. Newman, D.J. and Byrnes, J.S., The L^* Norm of a Polynomial with Coefficients ± 1 , submitted.
19. Rudin, W., Some Theorems on Fourier Coefficients, Proc. Amer. Math. Soc. 10 (1959), 855-859.
20. Salem and Zygmund, Some Properties of Trigonometric Series Whose Terms Have Random Signs, Acta Mathematica, Vol. 91, 1954.
21. Schroeder, M.R., Number Theory in Science and Communication (Second Ed.), Springer-Verlag, Berlin, 1985.
22. Shapiro, H.S., Extremal Problems for Polynomials and Power Series, Thesis for S.M. degree, Mass. Inst. of Tech. (1951).
23. Zygmund, A., Trigonometric Series, 2nd rev. ed., Cambridge University Press, 1959.

APPENDIX A
Additional Information

A-I. TECHNICAL PUBLICATIONS

1. Byrnes, J.S., The Minimax Optimization of an Antenna Array Employing Restricted Coefficients, Scientia (to appear, 1987).
2. Newman, D.J. and Byrnes, J.S., The L^{∞} Norm of a Polynomial with Coefficients ± 1 , (submitted).
3. Byrnes, J.S. and Newman, D.J., Null Steering Employing Polynomials with Restricted Coefficients, (submitted).
4. Byrnes, J.S., A Notched Filter Employing Coefficients of Equal Magnitude, (submitted).

A-II. PROFESSIONAL PERSONNEL

1. James S. Byrnes, Principal Investigator. President, Prometheus Inc. Ph.D. (Mathematics), Yeshiva University, 1967.
2. Donald J. Newman, Principal Scientist. Ph.D. (Mathematics), Harvard University, 1953.
3. Stephen Boyd, Senior Scientist. Ph.D. (Electrical Engineering), University of California at Berkeley, 1985.
4. Andre Giroux, Senior Scientist. Ph.D. (Mathematics), University of Montreal, 1973.
5. Martin Goldstein, Senior Scientist. Ph.D. (Mathematics), University of Wisconsin, 1969.

A-III. INTERACTIONS

I. Invited Papers Presented by J. S. Byrnes.

1. Department of Electrical Engineering, Stanford University. 7 October, 1986.
2. Naval Research Laboratory, Washington, DC, 10 November 1986.
3. Naval Underwater Systems Center, New London, CT.. 28 January 1987.

II. Upcoming Invited Lectures on this Work, by J. S. Byrnes.

1. Department of Mathematics, University of Maryland, April, 1987.
2. NATO Advanced Study Institute on EM Modeling and Measurements for Analysis and Synthesis, Tuscany, Italy. August, 1987.
3. XXIInd General Assembly of the International Union of Radio Science, Tel Aviv, Israel, September, 1987.

III. Consultations on Potential Air Force and Navy Applications of Prometheus Ideas.

1. Lecture presented by Donald J. Newman at Raytheon Equipment Division, Wayland, MA, 26 September 1986. Also, Newman and Byrnes of Prometheus consulted with Duane Matthiesen, Herb Groginsky, Leon Wardle, and Fred Daum of Raytheon, Hans Steyskal and John K. (Jay) Schindler of RADC - Hanscomb, Len Smith of MITRE, and Richard Turyn of GTE - Natick.
2. Stanford University, 7 October 1986. Byrnes and Boyd consulted with Professor Bernard Widrow.
3. Naval Research Laboratory, 10 November 1986. Byrnes consulted with Emanuel Vegh, Paul Crepeau, Frank Kretschmer, Karl Gerlach, J. Rao, Jack Ahern, S.N. Samaddar, and Allen Miller.
4. Various Locations, 15-17 December 1986. See attached Trip Report.
5. Naval Underwater Systems Center, 28 January 1987. Byrnes consulted with Roger Dwyer, John Fay, Kurt Hafner, Huck Quazi, Ed Eby, and Cliff Carter.

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TRIP REPORT

In connection with contract # F49620-86-C-0088
NULL STEERING APPLICATIONS OF POLYNOMIALS
WITH UNIMODULAR COEFFICIENTS
James S. Byrnes, Principal Investigator

A Phase I SBIR Contract with the
Air Force Office of Scientific Research
Dr. Arje Nachman, Program Manager

Dates of Trip: 15-17 December, 1986

Prometheus Personnel: James S. Byrnes, Principal Investigator
Donald J. Newman, Principal Scientist
Stephen Boyd, Senior Scientist

Installations Visited:

1. Rome Air Development Center, Hanscomb Air Force Base, MA (Byrnes, Newman, Boyd).
2. MIT Lincoln Labs, Bedford, MA (Byrnes, Newman, Boyd).
3. Raytheon Equipment Division, Wayland, MA (Byrnes, Newman).
4. MITRE, Bedford, MA (Byrnes, Newman).
5. Raytheon Submarine Signal Division, Portsmouth, RI (Byrnes).

Purposes of trip: To discuss new results obtained by Prometheus under the above contract, to learn of possible applications of these results at the above installations, and to obtain suggestions for directions of future research.

Personnel Visited:

1. RADC: Dr. Robert Mailloux (primary point of contact), Dr. Hans Steyskal (617-377-2052), Dr. Robert Shore, Mr. Jeff Herd.
2. MIT Lincoln Labs: Dr. Charles Rader (617-863-5500, x2574).
3. Raytheon-Wayland: Dr. Eli Brookner (primary point of contact, 617-358-2721, x5636), Dr. James Mullen, Mr. Fred Daum.
4. MITRE: Dr. Dean Carhoun (primary point of contact, 617-271-2518), Dr. Irving Reed, Dr. Warren Wilson (617-271-3913), Dr. John Cozzens (617-271-3404), Mr. Len Smith (617-271-3905), several others.
5. Raytheon-Portsmouth: Dr. Dave DeFanti (primary point of contact, 401-847-8000, x4411), Dr. Stan Chamberlain, Dr. Roger Pridham, Mr. Al Gerheim.

Results: Most of the above individuals found the Prometheus work very interesting, and had many suggestions for applications and follow-on work. These suggestions have been incorporated into our proposal "Polynomials with Restricted Coefficients and their Applications," and will form an integral part of our Phase II proposal.

7 January 87
Date

James S. Byrnes
James S. Byrnes, President

APPENDIX B

The L^4 Norm of a Polynomial With Coefficients ± 1

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A classic unresolved question regarding n -th degree polynomials with coefficients ± 1 is whether the maximum modulus of such a polynomial on the unit circle can be $n^{1/2} + o(n^{1/2})$. As shown by Kahane [2], if complex coefficients of modulus 1 are allowed then not only is it possible for this property to be satisfied, but the minimum modulus can be $n^{1/2} + o(n^{1/2})$ as well. Specifically, Kahane proved that for any n there is a polynomial of degree n with coefficients of modulus one whose modulus everywhere on the unit circle is $n^{1/2} + O(n^{3/4} \log n)$.

Erdos [1] had conjectured the existence of a $c > 0$ such that, for any polynomial P of the types described, $\|P\|_{\infty} > (1+c)n^{1/2}$. Clearly the Kahane result disproved this conjecture for the modulus 1 case, but the situation for coefficients ± 1 remains open. Employing an elegant construction Shapiro [4,3] demonstrated the achievability of the order of magnitude $n^{1/2}$, but the maximum modulus of the Shapiro polynomials is $(2n)^{1/2}$. Motivated by these considerations we examine the L^4 norm of such polynomials. As one might expect, this leads to several interesting combinatorial questions. We provide answers to some of these, and conclude with a refined version of the Erdos conjecture.

Throughout the paper n will be a positive integer, $P(z)$ will denote a polynomial of degree $n-1$ with coefficients ± 1 , and z will lie on the unit circle. Thus,

$$P(z) = \sum_{k=0}^{n-1} \epsilon_k z^k, \text{ each } \epsilon_k = 1 \text{ or } -1, z = e^{2\pi i \theta}, 0 \leq \theta < 1.$$

All integrals will be over $\theta \in [0, 1]$. We begin with a Lemma.

Lemma $\|P\|_{L^4}^4 = \sum_{\substack{j+k=l+m \\ 0 \leq j,k,l,m < n}} \epsilon_j \epsilon_k \epsilon_l \epsilon_m$

Proof $\|P\|_{L^4}^4 = \int_0^1 |P(e^{2\pi i \theta})|^4 d\theta$
 $= \int \left(\sum_{j=0}^{n-1} \epsilon_j z^j \right) \left(\sum_{k=0}^{n-1} \epsilon_k z^k \right) \left(\sum_{l=0}^{n-1} \epsilon_l z^{-l} \right) \left(\sum_{m=0}^{n-1} \epsilon_m z^{-m} \right)$
 $= \int \text{constant terms.}$

Since a constant term occurs in this product if and only if $j+k=l+m$, the result follows immediately.

Of interest is the expected value $E(\|P\|_{L^4}^4)$, if the coefficients ϵ_j are chosen at random.

Theorem 1 $E(\|P\|_{L^4}^4) = 2n^2 - n.$

Proof Clearly if exactly 3 of the indices j, k, l, m are identical, or if at least 3 of them are different, then $E(\epsilon_j \epsilon_k \epsilon_l \epsilon_m) = 0$. It therefore follows from the lemma that

$$E(\|P\|_{L^4}^4) = \sum_{\substack{j=l \text{ and } k=m \\ \text{or } j=m \text{ and } k=l}} \epsilon_j \epsilon_k \epsilon_l \epsilon_m. \quad (1)$$

For each of the $n(n-1)/2$ pairs of integers p, q , $0 \leq p < q < n$,

there are 4 terms appearing in (1), namely $\xi_p \xi_q \xi_p \xi_q$, $\xi_q \xi_p \xi_q \xi_p$, $\xi_p \xi_p \xi_q \xi_q$, $\xi_q \xi_q \xi_p \xi_p$. The only other terms in (1) are ξ_p^2 , $0 \leq p < n$. Since all of these terms equal 1,

$$E(\|P\|_{L^4}^4) = n + 4(n)(n-1)/2 = 2n^2 - n,$$

completing the proof of Theorem 1.

We now observe the improvement that is achieved when this random choice is replaced by the Shapiro coefficients.

Theorem 2 If $n=2^k$ and $P(z)$ is the Shapiro polynomial of degree $n-1$, then

$$\|P\|_{L^4}^4 = (4n^2 - (-1)^k n) / 3.$$

Proof Shapiro's polynomials are defined, together with his auxiliary polynomials Q , by the recurrence formulas

$$P_0(z) = Q_0(z) = 1,$$

$$P_{m+1}(z) = P_m(z) + z^{2^m} Q_m(z), \quad (2)$$

$$Q_{m+1}(z) = P_m(z) - z^{2^m} Q_m(z), \quad m \geq 0.$$

As a result,

$$|P_{m+1}(z)|^2 + |Q_{m+1}(z)|^2 = 2(|P_m(z)|^2 + |Q_m(z)|^2)$$

so that, as is well known,

$$|P_m(z)|^2 + |Q_m(z)|^2 = 2^{m+1}. \quad (3)$$

Now (2) and (3) yield

$$|P_{m+1}(z)|^2 = 2^{m+1} + 2 \operatorname{Re} \left(z^{2^m} Q_m(z) \bar{P}_m(z) \right). \quad (4)$$

Next we observe that $z^{2m} Q_m \bar{P}_m$ is composed solely of frequencies which are positive powers of z , so that it can be thought of as $Q_m \tilde{P}_m$, where \tilde{P}_m is the "reversed" polynomial of P . Thus

$$\int |P_{m+1}|^4 = 2^{2m+2} + 4 \int (\operatorname{Re} Q_m \tilde{P}_m)^2.$$

Since $f \equiv Q\tilde{P}$ is analytic and 0 at the origin,

$$0 = \int f^2 = \operatorname{Re} \left(\int f^2 \right) = \int (\operatorname{Re} f)^2 - \int (\operatorname{Im} f)^2,$$

so that

$$\int (\operatorname{Re} f)^2 = \frac{1}{2} \int |f|^2 = \frac{1}{2} \int |Q_m|^2 |\tilde{P}_m|^2 = \frac{1}{2} \int |P_m|^2 |Q_m|^2.$$

Altogether then, we have

$$\begin{aligned} \int |P_{m+1}|^4 &= 2^{2m+2} + 2 \int |P_m|^2 (2^{m+1} - |P_m|^2) \\ &= 2^{2m+3} - 2 \int |P_m|^4. \end{aligned} \quad (5)$$

The remainder of the proof is now simply induction on k . The result is obviously true for $k=0$, since $P_0(z)=1$. Furthermore, from (5) and the inductive hypothesis

$$\int |P_k|^4 = \frac{2^{2k+2} - (-2)^k}{3},$$

it follows that

$$\int |P_{k+1}|^4 = 2^{2k+3} - 2 \frac{2^{2k+2} - (-2)^k}{3} = \frac{2^{2(k+1)+2} - (-2)^{k+1}}{3}, \text{ as required.}$$

This completes the proof of Theorem 2.

Note that Theorem 2 implies that the L^4 norm of the $n-1$ st degree Shapiro polynomial is asymptotic to \sqrt{n} times the fourth root of $4/3=1.07457\sqrt{n}$. Based upon extensive numerical evidence employing the Bose-Einstein statistics methodology of Statistical Mechanics, we conjecture that the Shapiro polynomials do not give the minimum L^4 norm among all polynomials of the same degree with coefficients ± 1 , but that this minimum L^4 norm is asymptotically \sqrt{n} times the fourth

root of $6/5 \approx 1.04664\sqrt{n}$. Observe that the truth of this conjecture would imply that of the Erdos conjecture mentioned earlier, with $c=(6/5)^{1/4}-1 \approx .04664$.

References

1. Erdos, Paul, Some Unsolved Problems, Mich. Math. J. 4(1957), 291-300.
2. Kahane, Jean-Pierre, Sur les Polynomes a Coefficients Unimodulaires, Bull. London Math Soc. 12 (1980), 321-342.
3. Rudin, W., Some Theorems on Fourier Coefficients, Proc. Amer. Math. Soc. 10 (1959), 855-859.
4. Shapiro, H.S., Extremal Problems for Polynomials and Power Series, Thesis for S.M. degree, Mass. Inst. of Tech. (1957).

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APPENDIX C

Null Steering Employing Polynomials with Restricted Coefficients

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ABSTRACT

Several new designs of analytic null steering algorithms for linear arrays are described. Two of them, the \mathcal{A} -Technique and the Positive Coefficient Model, allow for placing an arbitrary number of nulls in arbitrary directions, while maintaining main beam and sidelobe level control. A method of incorporating these deterministic null steering techniques into existing adaptive algorithms is proposed. The resulting Direct Adaptive Nulling System offers the possibility of significant increases in array performance at very little cost.

Null Steering Employing Polynomials with Restricted Coefficients

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I. INTRODUCTION

In view of the well-known one-to-one correspondence between polynomials and linear arrays with commensurable separations between elements, as described in detail by Schelkunoff [9], null steering questions involving such arrays translate directly into mathematical problems regarding the locations of zeroes, on the unit circle, of polynomials. Furthermore, physical and electronic limitations placed upon the array elements, such as a maximum allowable power or a bound on the dynamic range, imply various restrictions upon the coefficients of these polynomials. Here, dynamic range refers to the ratio of the magnitudes of the largest to the smallest weight, or shading coefficient, of the array. Thus, the theoretically challenging question of the placement of zeroes at specified points on the unit circle, of polynomials whose coefficients satisfy certain restrictions, is also a problem of strong practical interest to antenna designers.

The design of filters is another application in which such questions arise. For example, the classical mathematical problem in notch filter design is to produce a polynomial whose magnitude on the unit circle is close to constant in almost all directions, but which has a small number (ie, 1, 2 or 3) of deep nulls ("notches") at specified points. In [5] the construction of [4] is employed to produce such a polynomial having one null, with the added feature that

all coefficients have the same magnitude. Hence, the dynamic range of the notch filters presented in [5] is one.

This paper addresses the null steering application described in the first paragraph. There are several factors which must be considered in the design of null steering algorithms. In addition to the basic problem of placing the nulls the main beam must be steered, the width of the main lobe controlled, and the sidelobe levels must be sufficiently below that of the main lobe. Control of the sidelobe level is usually achieved by attenuating the shading coefficients as one moves away from the center of the array. Often these attenuation factors (Chebyshev, Taylor, etc.) are chosen in advance, and may not be easily altered once the array is in place. This leads directly to a beautiful mathematical question, similar to the peak-factor problem in engineering attacked by Boyd [2], Schroeder [10] and others:

Given the magnitude of the coefficients of a polynomial P , a finite subset S of the unit circle C , and a point $p \in C$ distinct from those in S , choose the phases of these coefficients so that $P(z) = 0$ for all $z \in S$, the maximum on C of $|P(z)|$ occurs at $z = p$, and the maximum of $|P(z)|$ on a subset of C excluding an appropriate interval (the beamwidth) around p is as small as possible.

We consider various subproblems in this paper. Research on the general question is continuing.

II. DIRECT ADAPTIVE NULLING

Currently the most widely used class of null steering methods is known as adaptive nulling [1,3,6,7,8,12]. Adaptive arrays have

developed over the past twenty-five years as the preferred method of reducing the performance deterioration in signal reception systems which is inevitably caused by undesired noise entering the system. Sources for this noise include multipath affects, electronic countermeasures, clutter scatterer returns, antenna location errors, array element thermal noise, etc. The proliferation of such noise sources has greatly increased the importance of interference suppression in essentially all applications. Although such adaptive methods as the Widrow least mean squares (LMS) and Howells-Applebaum sidelobe canceller have achieved considerable success, difficult problems remain. Foremost among these are poor transient response, signal cancellation resulting from interaction between signal and interference, excessive computation time, and sidelobe degradation when jammer cancellation is attempted. A secondary problem is the lack of control in adaptive algorithms of the dynamic range of the weights.

These methods are indirect adaptive schemes; they do not explicitly form an estimate of the directions of arrival of interfering sources or explicitly steer nulls in those directions. A scheme in which these two tasks are actually performed can be called a direct adaptive algorithm. Thus, one approach to the solution of such problems is to complement an appropriate indirect adaptive algorithm with the analytic null steering methods described herein. In this way, the actual noise suppression achieved can be enhanced beyond that which would be available through either adaptive or analytic methods exclusively.

The first step in this "Direct Adaptive Nulling System" would be to employ available techniques, such as the maximum entropy method spectral estimation, a spatial discrete Fourier transform of the array outputs, a search in angle with an auxiliary beam [3], or the first loop of the indirect adaptive algorithm, to estimate noise and signal directions. Our analytic methods would then be applied to choose shading coefficients which place nulls in the estimated noise directions, while maintaining other desired properties of the beam pattern. Feeding these coefficients back into the adaptive portion of the algorithm then results in reestimates of the directions of the noises and signal, which are used in turn by the analytic portion to improve the choice of the shading coefficients. This process continues until convergence is achieved. Furthermore, each execution of the analytic step is essentially instantaneous, as the evaluation of the coefficients given the required directions is simply a matter of plugging the data into elementary formulas.

We expect such a direct scheme to outperform the indirect methods in cases where noise or interference is highly correlated with the desired signal, such as occurs with multipaths or intelligent jamming. Indirect schemes tend to perform poorly in these environments. In addition, a direct scheme allows much greater use of prior knowledge, such as known jammer locations or known multipaths. Thus, this interactive direct method offers the possibility of significant increases in performance at very little cost.

III. COEFFICIENTS OF EQUAL MAGNITUDE

An important subproblem of the general mathematical question

described earlier is the case when all of the coefficients of the polynomial have the same magnitude, which, by normalization, we can assume to be one. Such phase-only shading occurs, for example, in the design of transmitting arrays which are omnidirectional except for specified nulls. These features are crucial in certain communications areas, where it is desired to null out listeners in known directions while, at the same time, for maximum efficiency, all antenna elements are broadcasting at full power. Also, in order to minimize the relative size of the quantisation steps in a gradient algorithm such as LMS, the coefficient magnitudes should be kept as close as possible to unity [7, p. 153]. Note that this "equimagnitude" property of the coefficients precludes the use of attenuators, with the concomitant savings in electronic hardware.

The most elementary example of the above is the unshaded array - all coefficients are 1. In spite of its simplicity, this uniform array is of practical importance. Observe that in this case the zeroes of the polynomial are almost uniformly spaced around the unit circle, occurring at all of the $n+1$ th (where n is the degree of the polynomial) roots of unity except $z=1$, where there is a maximum.

At the other extreme is the case where an n -fold zero is required at one point. One application of this, as discussed by Steyskal [11], is to broaden a pattern null so as to null an entire sector. Clearly, by a simple change of variables, this zero-point can be assumed to occur at $z=1$. It is a straightforward matter to construct such a polynomial with coefficients of magnitude 1; in fact, the coefficients may all be taken to be ± 1 . Namely, define $P(z)$ by:

$$P(z) = \prod_{m=0}^{n-1} (1 - z^{2^m}) \quad (1)$$

The problem with this construction is that, although $P(z)$ obviously satisfies the required properties, it does so at very high cost. Since P has degree $2^n - 1$, its realization requires an array with 2^n elements. We show in the following theorem that, for all but small values of n , this situation may be greatly improved by allowing coefficients to be 0 as well as ± 1 . Since this simply means that some array elements are turned off, the dynamic range of the coefficients is not affected in any meaningful way.

Theorem I Let $n \geq 10$. Then there is a polynomial $P(z)$, of degree less than n^2 , such that $P(z)$ has an n -fold zero at $z=1$, and all the coefficients of P are either ± 1 or 0.

Proof of Theorem I Given $n \geq 10$, choose k so that

$$2^k > k n^2 \quad (2).$$

Since $x/\ln(x)$ is increasing for $x \geq e$, and $10/\ln(10) > 3/\ln(2)$, the choice $k = n^2$ certainly implies (2). Actually, for any $\epsilon > 0$, it is clear that for n large enough we may take $k = n^{2+\epsilon}$, but such precision is not necessary here. Now, for each arbitrary subset S of the set of nonnegative integers less than k , let

$$Q(z) = \sum_{m \in S} z^m,$$

and form the vector

$$(Q(1), Q'(1), Q''(1)/2!, \dots, Q^{(n-1)}(1)/(n-1)!).$$

These are integer vectors, and the largest entry is bounded by

$$\sum_{m=0}^{k-1} m^{n-1} \leq kn.$$

Thus, there are less than kn^2 such vectors. Since there are 2^k subsets S , and so 2^k polynomials Q , equation (2) implies that at least two distinct polynomials, say $Q_1(z)$ and $Q_2(z)$, have the same associated vector. Hence, $P(z) = Q_1(z) - Q_2(z)$ is the desired polynomial, and Theorem I is proven.

The idea underlying equation (1), which we call "encapsulation," may also be employed to construct polynomials with coefficients of magnitude 1 that place any number of arbitrary nulls. Namely, we have:

Theorem II For any positive integer n , let $\{z_m\}_{m=1}^n$ be an arbitrary set of (not necessarily distinct) points on the unit circle. Then there is a polynomial $P(z)$ with coefficients all of magnitude 1, of degree $2^n - 1$, satisfying $P(z_m) = 0$, $1 \leq m \leq n$.

Proof of Theorem II As indicated above, we simply produce an explicit formula for $P(z)$:

$$P(z) = \prod_{m=0}^{n-1} (z^{2^m} - z_m^{2^m}).$$

It is straightforward to see that this $P(z)$ satisfies the required properties, establishing Theorem II.

IV. SIMULTANEOUS NULL STEERING AND MAIN BEAM PLACEMENT

In the previous section, we attacked the subproblem of the general question stated earlier which arises when the dynamic range of the coefficients is required to be one. We now consider another aspect of the original problem, perhaps of more interest to antenna designers. Namely, how can arbitrary nulls be placed while maintaining a specified main beam direction and specified maximum sidelobe level? We describe two methods, the β -Technique and the Positive Coefficient Model, of achieving these goals.

To set the problem, again let n denote any positive integer, and let $S = \{z_m\}_{m=1}^n$ be an arbitrary set of (not necessarily distinct) points on the unit circle. Also, let z_0 be a point on the unit circle distinct from those in S . Our methods allow the placement of zeroes of a polynomial P at all points in S , while simultaneously having the maximum of $|P|$ on the unit circle occur at $z=z_0$. Furthermore, the difference between $|P(z_0)|$ and the highest sidelobe can be made arbitrarily large. As will be seen, the costs encountered in achieving the last property are an increase in the degree of P , and a loss of control of the dynamic range of the coefficients.

To proceed with the constructions, define the angles $\{\theta_m\}$, $0 \leq m \leq n$, by $z_m = \exp(i\theta_m)$, $-\pi \leq \theta_m \leq \pi$. As before, a simple change of variables allows us to assume $\theta_0 = 0$, so that $z_0 = 1$.

Method 1. The β -Technique

Let $\beta = -\cot^{-1} 2 \sum_{m=1}^n \cot(\theta_m/2)$, $z^* = \exp(i\beta)$,

and define $Q(z)$ by

$$Q(z) = (z - z^*) \prod_{m=1}^n (z - z_m).$$

A straightforward calculation shows that $|Q(z)|$ has a relative maximum at $z=1$. Hence, for c a large enough positive integer, $P(z) = (1+z)^c Q(z)$ will certainly satisfy the required properties. It can be shown that, in order to guarantee an absolute maximum of $|P(z)|$ at $z=1$, it is sufficient to take $c = \lceil 1/\epsilon \rceil$, where $\epsilon = \min |\theta_m|$. Of course, in order to further increase the main lobe level relative to the sidelobes, it will be necessary to take c larger.

Method II. The Positive Coefficient Model

For each m , $1 \leq m \leq n$, choose the smallest positive integer k_m such that $\exp(ik_m \theta_m)$ lies in the left half plane, and define $P(z)$ by

$$P(z) = \prod_{m=1}^n (z^{k_m} - z_m^{k_m}) (z^{k_m} - \bar{z}_m^{k_m}).$$

Clearly $P(z)$ has the necessary zeroes. Furthermore, all of the coefficients of P are positive, so that the maximum of $|P(z)|$ on the unit circle obviously occurs at $z=1$. Once again, it is simple to further increase the main lobe level relative to the sidelobes by multiplying $P(z)$ by an appropriate positive integer power of $1+z$.

There are two additional points which can be made about the Positive Coefficient Model. One is that its electronic implementation will be greatly simplified as compared to that of arbitrary shading coefficients, since the positivity eliminates the need for phase

shifters. A second is that some control of the dynamic range of the coefficients can be achieved by combining this method with the encapsulation technique discussed earlier, if we again ignore the effects of 0 coefficients.

V. CONCLUSION

Various aspects of a fascinating problem in classical mathematical analysis, with direct applications to antenna array design, have been discussed, and several results obtained. Foremost among these are two analytic methods for placing an arbitrary number of nulls in arbitrary directions, while maintaining main beam and sidelobe level control. A method of incorporating these analytic null steering techniques into existing adaptive algorithms is proposed. The resulting Direct Adaptive Nulling System offers the possibility of significant increases in array performance at very little cost.

BIBLIOGRAPHY

1. Baird, C.A. and Rassweiler, G.G., Adaptive Sidelobe Nulling Using Digitally Controlled Phase-Shifters, IEEE Trans. Antennas & Propagation AP-24, no. 5 (1976), 638-649.
2. Boyd, Stephen, Multitone Signals with Low Crest Factor, IEEE Trans. Cir. & Systems 33 (1986), 1018-1022.
3. Brookner, Eli and Howell, James M., Adaptive-Adaptive Array Processing, IEEE Proc. 74 (1986), 602-604.
4. Byrnes, J.S., On Polynomials with Coefficients of Modulus One, Bull. London Math. Soc. 9 (1977), 171-176.
5. Byrnes, J.S., A Notch Filter Employing Coefficients of Equal Magnitude, (in preparation).
6. Griffiths, L.J., A Simple Adaptive Algorithm for Real-Time Processing in Antenna Arrays, Proc. IEEE 57 (1969), 1696-1704.
7. Hudson, J.E., Adaptive Array Principles, Peter Peregrinus Ltd., New York, 1981.
8. Monzingo, Robert A. and Miller, Thomas W., Adaptive Beamforming, Wiley, New York, 1980.
9. Schelkunoff, S.A., A Mathematical Theory of Linear Arrays, Bell System Tech. J.39 (1960), 80-107.
10. Schroeder, M.R., Synthesis of Low-Peak-Factor Signals and Binary Sequences with Low Autocorrelation, IEEE Trans. Inf. Th. 16 (1970), 85-89.
11. Steyskal, Hans, Synthesis of Antenna Patterns with Prescribed Nulls, IEEE Trans Ant. & Prop. 30 (1982), 273-279.
12. Widrow, B. et al, Adaptive Antenna Systems, Proc. IEEE 55, No. 12 (1967), 2143-2159.

APPENDIX D

The Minimax Optimization of an Antenna Array Employing Restricted Coefficients

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It is well known that the determination of optimal shading coefficients for an antenna array, using various measures of optimality, is a computational problem of order n^3 , where n is the number of array elements. Consequently, if it is possible to achieve optimum results by employing real shading coefficients, so that the effective number of coefficients to calculate is halved, the computational load can be reduced by a factor of 8. It is shown by Lewis and Streit [1] that real coefficients do indeed suffice for the minimax design of a linear antenna array. The purpose of this note is to prove that a similar result can be achieved, under certain circumstances, for a conformal array.

To precisely define the problem, assume that there are n elements located at points (x_j, y_j, z_j) , $1 \leq j \leq n$, and a set F of farfield points $u = (\cos\alpha, \cos\beta, \cos\gamma)$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Let the shading vector be $w = (w_1, w_2, \dots, w_n)$, and denote the beam pattern by

$$T(u, w) = \sum_{j=1}^n w_j \exp \left(i \frac{2\pi}{\lambda} (x_j \cos\alpha + y_j \cos\beta + z_j \cos\gamma) \right). \quad (1)$$

where λ is the wavelength of the design frequency. The minimax optimization problem can be stated as follows:

(*) Choose shading vector w so as to minimize the quantity $\max_{u \in F} |T(u, w)|$ subject to the normalization

$$T(u_0, w) = 1. \quad (2)$$

Here, $u_0 = (\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$ is a fixed farfield point. The normalization (2) is intended to force the Maximum Response Axis (MRA) to occur at the farfield point u_0 . Consequently u_0 should not lie in the set F .

It is proven herein that in certain cases problem (*) can be solved, with the same minimum achieved, when the weight vector w is restricted to be real. Namely, the following theorem holds:

Theorem 1. Let F be symmetric with respect to the origin of (x,y,z) space; that is, $u \in F$ if and only if $-u \in F$. Let the positive z -axis be the MRA, and assume that all z_j are integer multiples of $\lambda/2$. Then the minimum achieved in solving problem (*), when the weight vector w is restricted to be real, is equal to the minimum achieved without this restriction.

Proof. Let $v=(v_1, v_2, \dots, v_n)$ be any (complex, in general) vector which solves problem (*), and let $v_j = r_j + is_j$. The proof will be finished if it can be shown that the (real) vector $r=(r_1, r_2, \dots, r_n)$ also solves (*). Toward this end, first consider the normalization constraint (2). Since the MRA is the positive z -axis,

$$\alpha_0 = \beta_0 = \pi/2 \text{ and } \gamma_0 = 0.$$

Combining this with (1) and (2) yields

$$1 = T(u_0, v) = \sum_{j=1}^n v_j \exp \left(i \frac{2\pi}{\lambda} z_j \right), \quad (3)$$

so that

$$\sum_{j=1}^n r_j \cos \left(\frac{2\pi}{\lambda} z_j \right) - s_j \sin \left(\frac{2\pi}{\lambda} z_j \right) = 1, \quad (4)$$

$$\sum_{j=1}^n r_j \sin \left(\frac{2\pi}{\lambda} z_j \right) + s_j \cos \left(\frac{2\pi}{\lambda} z_j \right) = 0.$$

However, since all z_j are a multiple of $\lambda/2$, it follows that all terms in (4) involving $\sin(2\pi z_j/\lambda)$ are zero. When combined with (3) this immediately implies that

$$T(u_0, r) = \sum_{j=1}^n r_j \exp \left(i \frac{2\pi}{\lambda} z_j \right) = 1, \quad (5)$$

which means that the vector r satisfies the normalization constraint also.

To see that r satisfies the required minimax property, recall that F is symmetric with respect to the origin, and apply the method of [1]. Thus, letting an overbar denote complex conjugate, any weight vector w must satisfy

$$\max_{u \in F} |T(u, \bar{w})| = \max_{u \in F} |T(-u, \bar{w})| = \max_{u \in F} |T(u, w)|,$$

so that

$$\begin{aligned} \max_{u \in F} |T(u, \operatorname{Re}(w))| &= \max_{u \in F} \left| T(u, \frac{1}{2} w + \frac{1}{2} \bar{w}) \right| \\ &\leq \frac{1}{2} \max_{u \in F} |T(u, w)| + \frac{1}{2} \max_{u \in F} |T(u, \bar{w})| = \max_{u \in F} |T(u, w)|. \end{aligned} \quad (6)$$

Applying (6) to the optimum vector v yields

$$\max_{u \in F} |T(u, r)| \leq \max_{u \in F} |T(u, v)|. \quad (7)$$

However, since v minimizes this maximum among all normalized weight vectors, and because r satisfies (2), it follows that the inequality in (7) must in fact be an equation. Thus the real vector r solves problem (*), and the theorem is proven.

We now show that the above method also yields a more general result for a linear array than that in [1]. Namely, we prove that the shading coefficients in a minimax optimization problem for a linear array can be taken to lie along any fixed line L through the origin in the complex plane, if the normalization condition is altered to require that the sum of the weights be either of the two fixed complex numbers of modulus one lying on L . Thus, consider n omnidirectional elements located at arbitrary fixed positions x_k along the x -axis. The minimax problem for a linear array steered broadside becomes:

(**) Choose the complex weight vector $w = (w_1, w_2, \dots, w_n)$ so as to minimize the quantity

$$\text{Max}_{u, |u| \leq 1} |T(u)|,$$

subject to the normalization constraint

$$T(0) = \sum_{k=1}^n w_k = 1. \quad (8)$$

In this case, the beam pattern $T(u)$ is

$$T(u) = \sum_{k=1}^n w_k \exp(-id_k u), \text{ with}$$

$$d_k = 2\pi \frac{x_k}{\lambda}, \quad u = \sin \theta,$$

θ = directional of arrival of a plane wave of wavelength λ , and u_0 is a small positive number.

As indicated above, Lewis and Streit show that problem (**) has a solution with a real weight vector w . Our generalization of their result is as follows:

Theorem 2. Replace the normalization (8) by

$$T(0) = \sum_{k=1}^n w_k = \exp(i\Upsilon). \quad (9)$$

Then problem (**) has a solution where all w_k are taken along the line given (in polar coordinates) by $\theta = \Upsilon$.

Proof. The projection of each component w_k of any weight vector w upon the line $\theta = \Upsilon$ is given by

$$\exp(i\Upsilon) (\cos \Upsilon \operatorname{Re} w_k + \sin \Upsilon \operatorname{Im} w_k),$$

so that the method employed to obtain (7) yields

$$\text{Max}_{u, |u| \leq 1} \left| \sum_{k=1}^n \exp(i\Upsilon) (\cos \Upsilon \operatorname{Re} w_k + \sin \Upsilon \operatorname{Im} w_k) \exp(-id_k u) \right|$$

$$\leq \frac{1}{2} \text{Max}_u \left| \sum_{k=1}^n w_k \exp(-id_k u) \right| + \frac{1}{2} \text{Max}_u \left| \sum_{k=1}^n \bar{w}_k \exp(-id_k u) \right| = \text{Max}_u \left| \sum_{k=1}^n w_k \exp(-id_k u) \right|.$$

Furthermore, it follows from (9) that

$$\sum_{k=1}^n \exp(i\psi) (\cos \psi \text{Re } w_k + \sin \psi \text{Im } w_k) = \exp(i\psi) \text{ as well.}$$

Now suppose that $\{v_k\}$ is any sequence of weighting coefficients satisfying this new constrained minimax problem. That is,

$$\sum_{k=1}^n v_k = \exp(i\psi)$$

and, for any $\{w_k\}_{k=1}^n$ satisfying

$$\sum_{k=1}^n w_k = \exp(i\psi),$$

$$\text{Max}_u \left| \sum_{k=1}^n v_k \exp(-id_k u) \right| \leq \text{Max}_u \left| \sum_{k=1}^n w_k \exp(-id_k u) \right|.$$

Then, by the above,

$$\left\{ \exp(i\psi) (\cos \psi \text{Re } v_k + \sin \psi \text{Im } v_k) \right\}_{k=1}^n$$

also satisfies the problem, and these coefficients all lie on the line $\theta = \psi$. This completes the proof of Theorem 2.

Finally, as Lewis and Streit note, under some circumstances a condition such as $T(\hat{u}) = 0$ for some point $\hat{u} \neq 0$ might be a crucial requirement added to problem (**). They correctly observe that now a solution with real coefficients need not necessarily exist. However, it is quite possible that a judicious choice of ψ in the above will allow for the solution of this modified problem to be found. We leave this as a subject for future research.

Reference

1. James T. Lewis and Roy L. Streit, "Real Excitation Coefficients Suffice for Sidelobe Control in a Linear Array", IEEE Transactions on Antennas and Propagation, Vol. AP-30, No. 6, 1982, 1262-1263.

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APPENDIX E

A Notch Filter Employing Coefficients of Equal Magnitude

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Abstract

A nearly ideal notch filter, employing coefficients of equal magnitude, is described. Applications to the design of transmitting antenna arrays are discussed briefly. The construction is based upon earlier work of the author involving polynomials with restricted coefficients.

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Figure Captions

- Figure 1. A graph of $|P(\theta)|$ for $N=10$.
- Figure 2. A graph of $|P(\theta)|$ for $N=60$.
- Figure 3. A graph of $|Q(\theta)|$ for $N=10$.
- Figure 4. A graph of $|Q(\theta)|$ for $N=60$.
- Figure 5. A graph of $|Q(\theta)|$ for $N=60$, in dB scale.

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Abstract

A nearly ideal notch filter, employing coefficients of equal magnitude, is described. Applications to the design of transmitting antenna arrays are discussed briefly. The construction is based upon earlier work of the author involving polynomials with restricted coefficients.

I. INTRODUCTION

The classical mathematical problem in notch filter design is to produce a polynomial whose magnitude on the unit circle is close to constant in almost all directions, but which has a small number (i.e., 1, 2 or 3) of deep nulls ("notches") at specified points. Such filters are applied, for example, to remove spectral lines from otherwise broadband spectra. In this paper, we produce such polynomials having one null, with the added feature that all coefficients have the same magnitude. For convenience, this magnitude is assumed to be one. Observe that this "unimodular" property allows

the direct application of these polynomials to the design of transmitting antenna arrays which are omnidirectional except for 1 null. This feature is crucial in certain communications areas, where it is desired to null out one listener in a known direction while at the same time, for maximum efficiency, all antenna elements are broadcasting at full power.

If the polynomial $P(z)$ is of degree $n-1$, it is clear from the Parseval Theorem that its L^2 norm (i.e., RMS value) is exactly $n^{1/2}$, since there are n coefficients each of magnitude 1. Thus, for $|P(z)|$ to be close to constant on $|z|=1$, that constant must be $n^{1/2}$. The question of the existence of such polynomials is a classic one in mathematical analysis. Its study was apparently initiated by Hardy [11,p.199], and furthered by Littlewood [8,9], Erdős [5], Newman [1,2,3,10] and others. A basic result concerning these problems was obtained by the author [4], which paved the way for solutions, by Körner [7] and Kahane [6], of two of the fundamental conjectures in this area. We modify the construction given in [4] to produce nearly ideals filters with one notch.

II. APPROACH AND RESULTS

Our starting point is the polynomial P , of degree N^2-1 , given by

$$P(\theta) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \exp(2\pi i j k N^{-1}) z^{j+kN}, \quad z = \exp(2\pi i \theta).$$

It is shown in [4] that $P(\theta)$ satisfies:

- (i) $|P(jN^{-1})| = N$ for all integers j ;
- (ii) For any ϵ , $N^{-1} \leq \epsilon \leq 1/2$, $|P(\theta)| = N + E$ for $\epsilon \leq \theta \leq 1/2$, where
 $|E| < 1 + 2\pi^{-1} + 5(\pi\epsilon)^{-1}$;
- (iii) $\min |P(\theta)| = O(1)$; and
- (iv) $|P(\theta)| < (2 + 3\pi^{-1})N + O(1)$ for all θ .

Recent numerical evidence suggests that (iv) can be strengthened to:

- (iv') $|P(\theta)| < 1.3N$ for all θ .

Figures 1 and 2, which show $|P(\theta)|$ as a function of θ for $N=10$ (i.e., P of degree 99) and $N=60$ (P of degree 3599), clarify the above properties. Thus, as $N \rightarrow \infty$, the magnitude of P is asymptotically close to constant except for the immediate neighborhood of one point. By a simple change of variables, it is clear that this special point can be taken anywhere on the unit circle.

If $P(\theta)$ is changed by removing the first N terms, all of whose coefficients are +1, and then dividing by z^N , there results

$$Q(\theta) = \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \exp(2\pi i j k N^{-1}) z^{j+(k-1)N}.$$

This Q , which is of degree $N^2 - N - 1$, is the desired modification of P . In fact, estimates (ii) and (iv') remain true for Q , while in addition

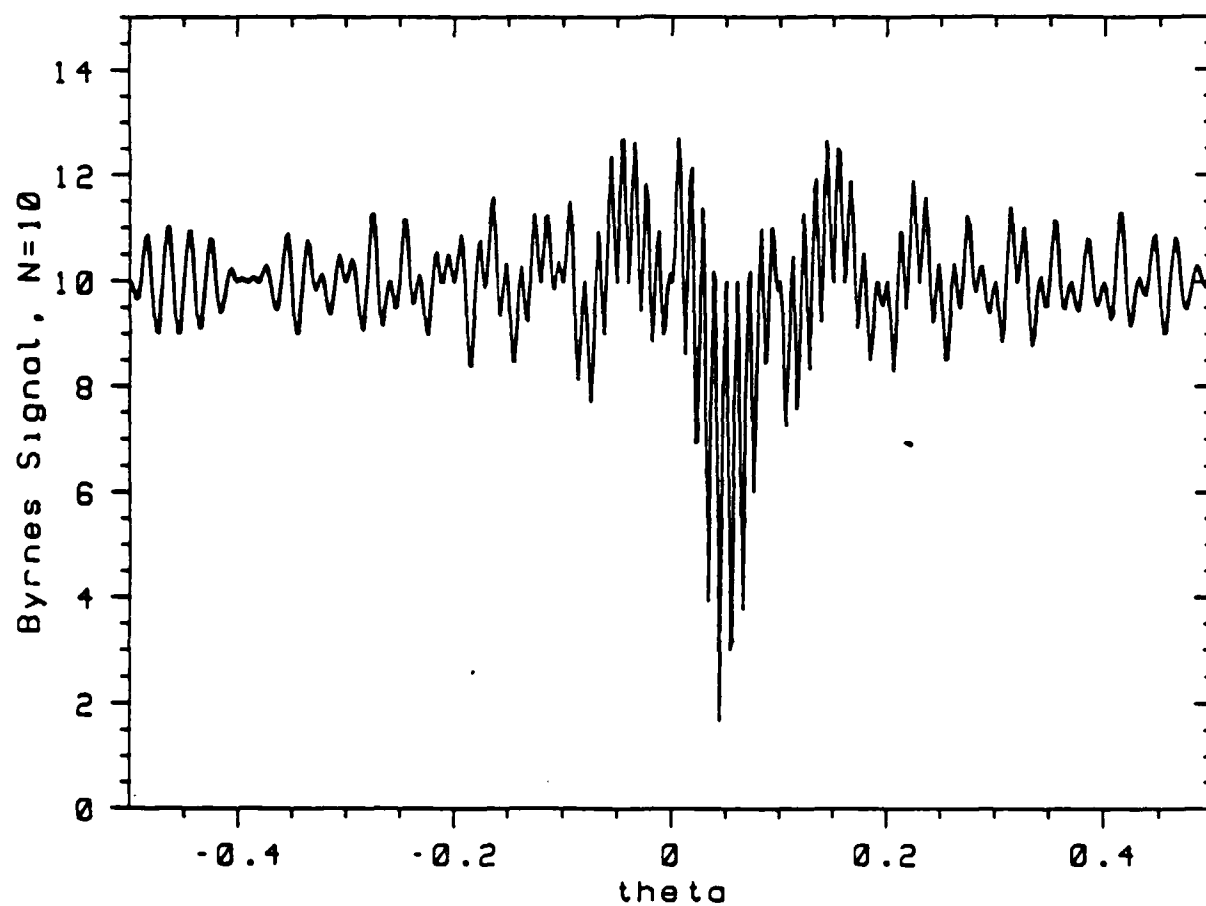


Figure 1. A graph of $|P(\theta)|$ for $N=10$.

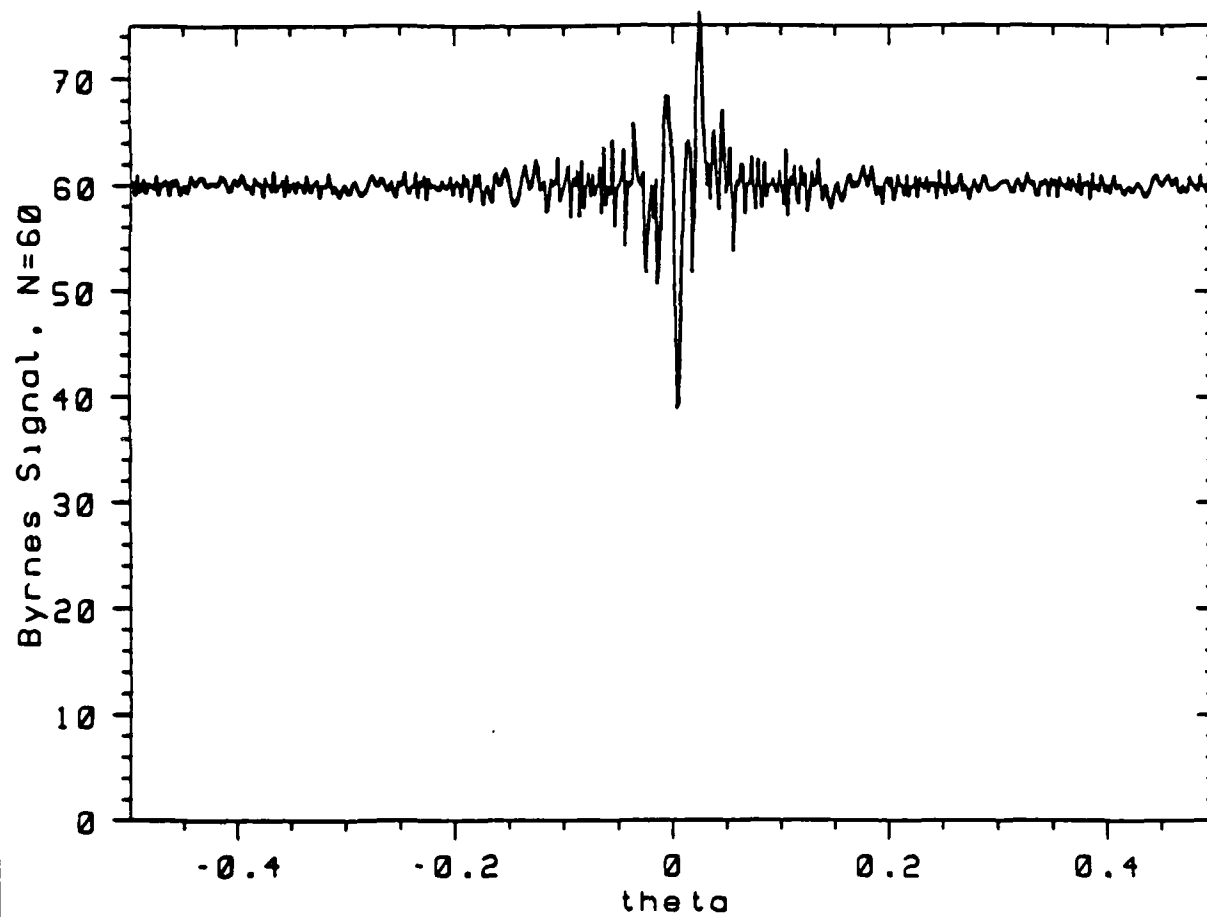


Figure 2. A graph of $|P(\theta)|$ for $N=60$.

$Q(0)=0$. Also, it can be shown that the null width of Q is less than $2/N$. Figures 3 and 4, which exhibit $|Q(\theta)|$ as a function of θ for $N=10$ (Q of degree 89) and $N=60$ (Q of degree 3539), and figure 5, which transforms the plot of figure 4 to a dB scale, show that Q is indeed the nearly ideal notch filter discussed earlier. Once again, it is clear that a change of variables allows the relocation of the notch to any desired θ .

BIBLIOGRAPHY

1. Beller, E. and Newman, D.J., An Extremal Problem for Polynomials, Proc. Amer. Math. Soc. 29 (1971), 474-481.
2. Beller, E. and Newman, D.J., An Extremal Problem for the Geometric Mean of Polynomials, Proc. Amer. Math. Soc. 39 (1973), 313-317.
3. Beller, E. and Newman, D.J., The Minimum Modulus of Polynomials, Proc. Amer. Math. Soc. 45 (1974), 463-465.
4. Byrnes, J.S., On Polynomials with Coefficients of Modulus One, Bull. London Math. Soc. 9 (1977), 171-176.
5. Erdos, Paul, Some Unsolved Problems, Mich. Math. J. 4(1957), 291-300.
6. Kahane, Jean-Pierre, Sur les Polynomes a Coefficients Unimodulaires, Bull. London Math. Soc. 12 (1980), 321-342.
7. Korner, T.W., On a Polynomial of Byrnes, Bull. London Math Soc. 12 (1980), 219-224.
8. Littlewood, J. E., On the Mean Values of Certain Trigonometric Polynomials, J. Lon. Math. Soc. 36 (1961), 307-334.

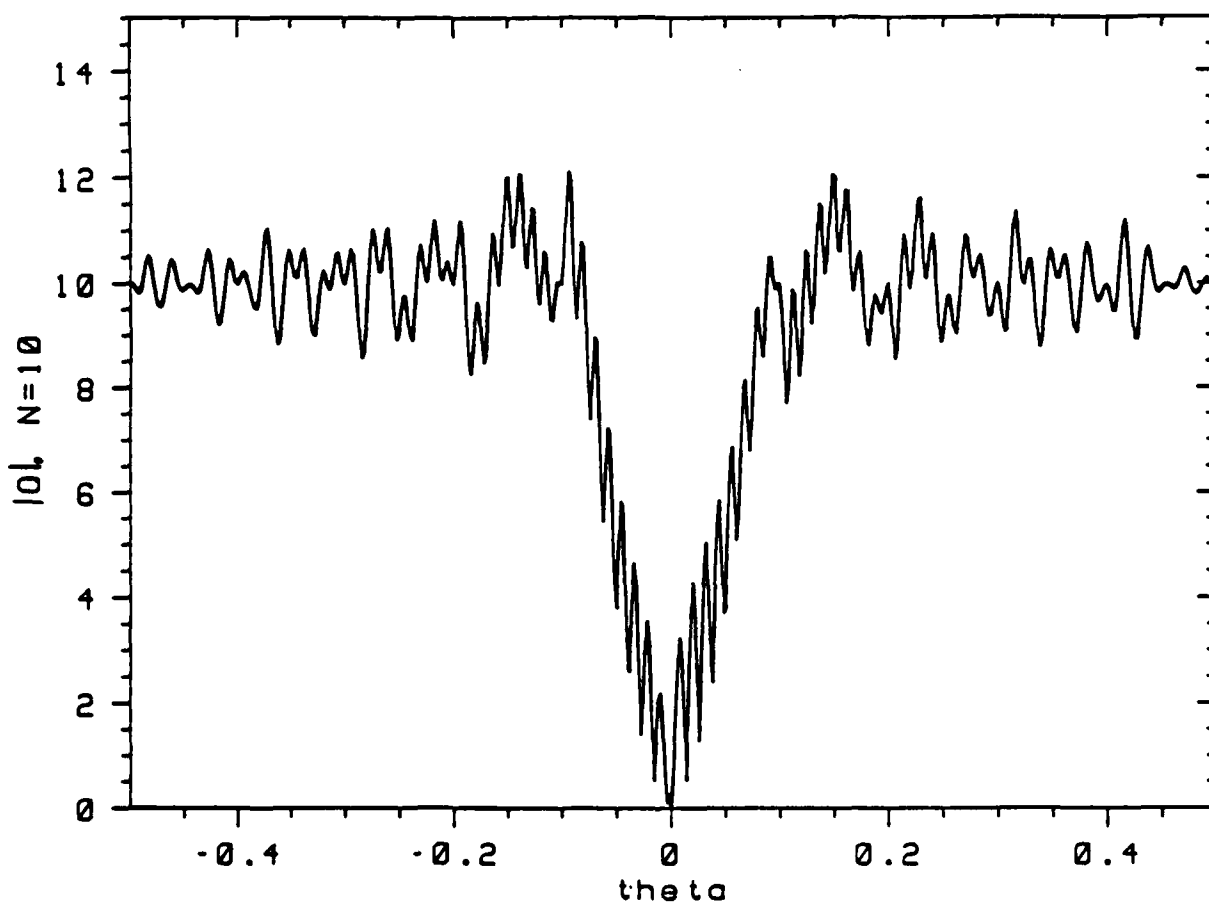


Figure 3. A graph of $|Q(\theta)|$ for $N=10$.

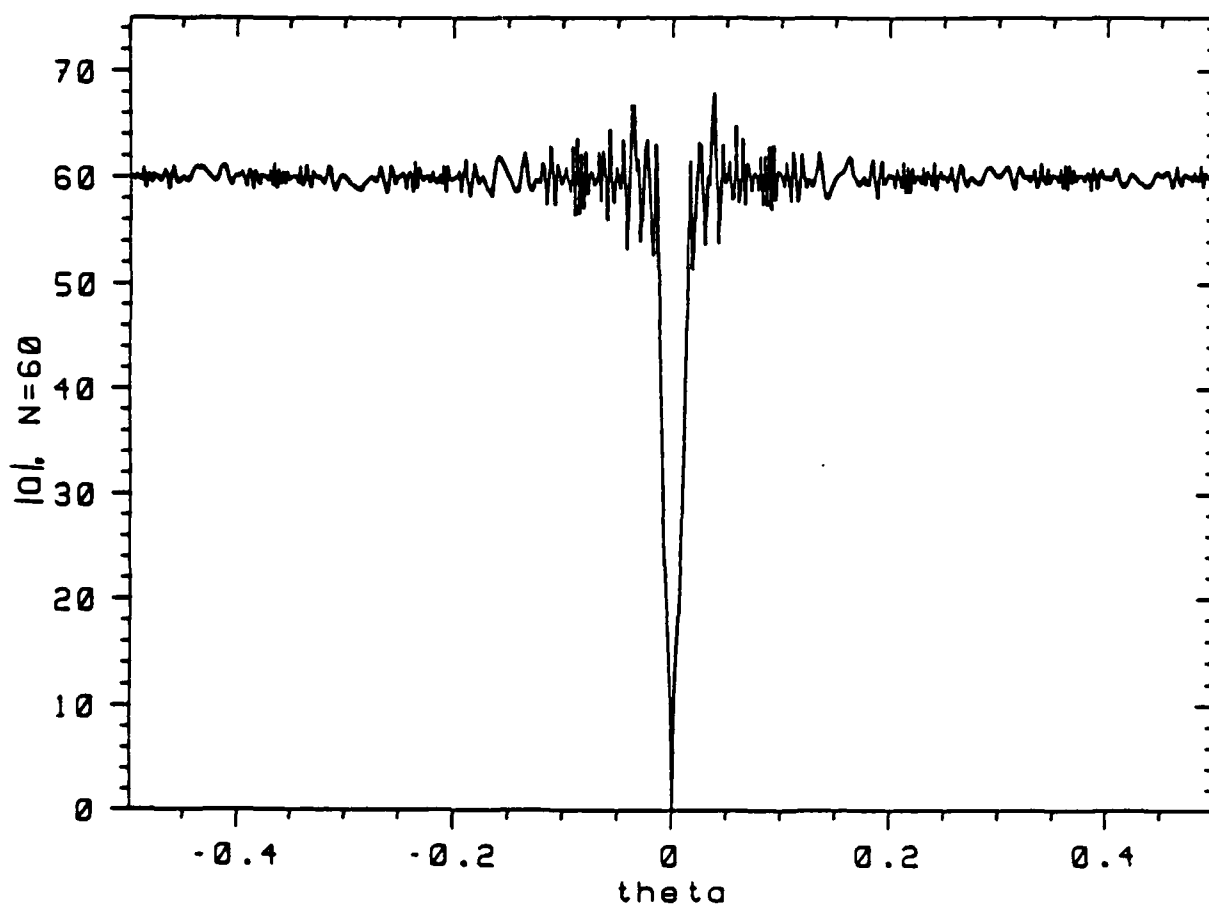


Figure 4. A graph of $|Q(\theta)|$ for $N=60$.

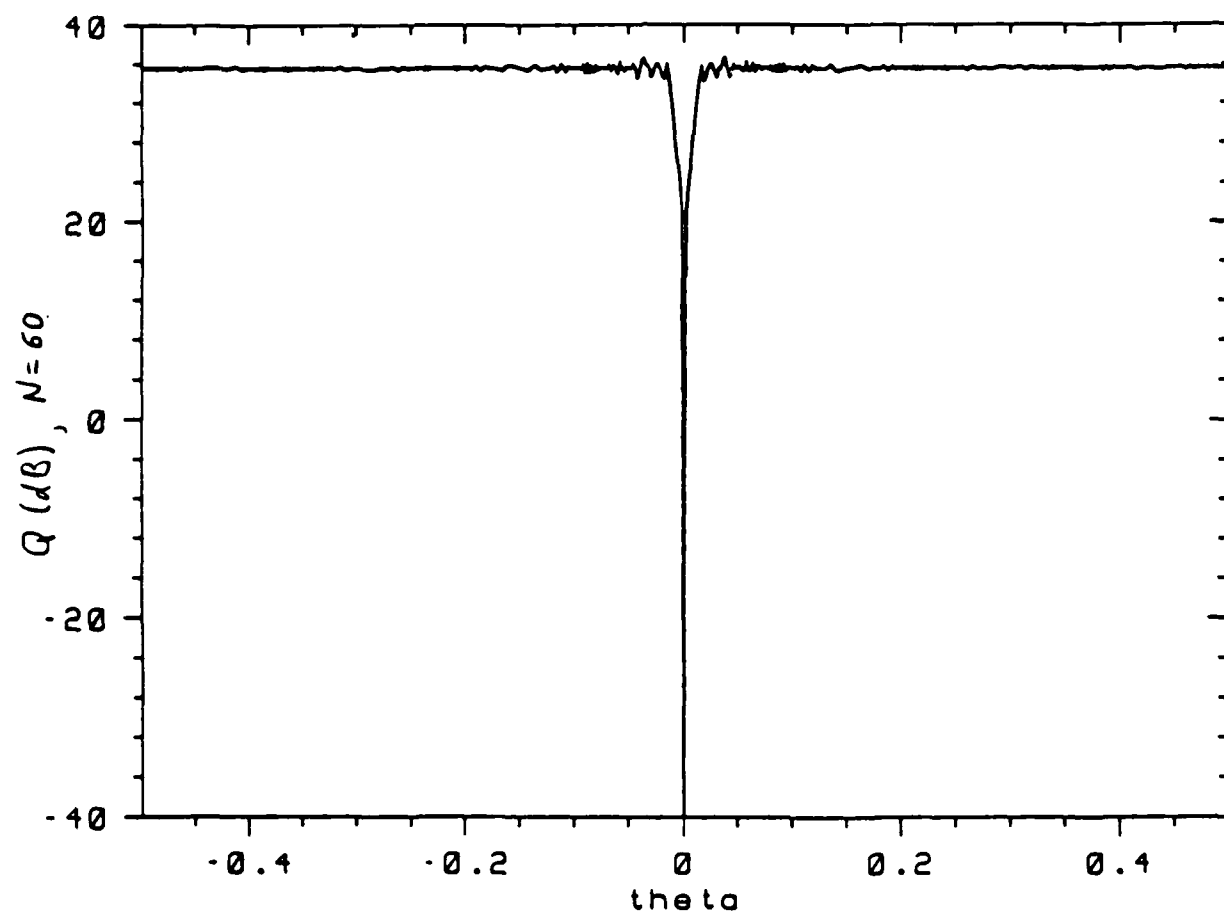


Figure 5. A graph of $|Q(\theta)|$ for $N=60$, in dB scale.

9. Littlewood, J. E., On the Mean Values of Certain Trigonometric Polynomials, II, Illinois J. Math. 6 (1962), 1-39.
10. Newman, D.J., An L^1 Extremal Problem for Polynomials, Proc. Amer. Math. Soc. 16 (1965), 1287-1290.
11. Zygmund, A., Trigonometric Series, 2nd rev. ed., Cambridge University Press, 1959.

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